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Mathematical Reviews

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NUMBER THEORY

Bell, E. T. A method in rational diophantine analysis. Proc. Nat. Acad. Sci. U. S. A. **30**, 355–359 (1944). [MF 11366]

By means of rational substitutions involving auxiliary Diophantine equations, the author is able to reduce rational Diophantine systems to those of multiplicative type. In this way he finds all or only some of the solutions of the original system according as the substitutions are or are not rationally reversible. The method is illustrated by means of two examples. The first is the classical ternary cubic leading to Ryley's theorem on the decomposition of every rational number as the sum of three rational cubes, while, in the second, the auxiliary equation reduces the given equation to one of multiplicative type by rendering the equation algebraically reducible. *I. A. Barnett.*

Uhler, H. S. First proof that the Mersenne number M_{157} is composite. Proc. Nat. Acad. Sci. U. S. A. **30**, 314–316 (1944). [MF 11359]

The present note gives details on the application of Lucas' test for the primality of 2^n-1 to the number $2^{157}-1$. As announced earlier [Mathematical Tables and Aids to Computation **1**, 333 (1944)], this number turns out to be composite as conjectured exactly 300 years ago by Mersenne. In fact the 156th term of the series 4, 14, 194, 37634, 1416317954, $\cdots$ is not divisible by $2^{157}-1$, there being a remainder of 118 57508 80382 71696 98184 73569 85091 23773 18030 92037. The author announces the beginning of a corresponding investigation of $2^{167}-1$. In a recent letter to the reviewer the author states that this number is likewise composite, confirming Mersenne's conjecture. There remain only four more Mersenne numbers to be tested: 2^n-1 for $n=193$, 199, 227, and 229.

D. H. Lehmer (Berkeley, Calif.).

Davenport, H. On the minimum of a ternary cubic form. J. London Math. Soc. **19**, 13–18 (1944). [MF 11314]

The author proves the following results related to work of Mordell [J. London Math. Soc. **17**, 107–115 (1942); these Rev. **4**, 131]. Let m be the minimum of $f=x^3+y^3+z^3$, where x, y, z are linear forms in the integers u, v, w (not all zero) with real coefficients whose determinant is 1. Then

$$m\leqq\frac{8}{\Gamma^3(4/3)\{2+27\sqrt{3}/(2\pi)\}}=1.1897\cdots.$$

There is a particular form given for which m is .8151.

B. W. Jones (Ithaca, N. Y.).

Mordell, L. J. On the minimum of a ternary cubic form. J. London Math. Soc. **19**, 6–12 (1944). [MF 11313]

If f in the above review is replaced by $F=x^3+y^3+z^3+Kxyz$, the author derives upper bounds for the minimum of F which are lower than ~~his previous results~~ those obtained by Minkowski. For $K=0$, Mordell's result is not as good as that of Davenport above.

B. W. Jones (Ithaca, N. Y.).

Mordell, L. J. Observation on the minimum of a positive quadratic form in eight variables. J. London Math. Soc. **19**, 3–6 (1944). [MF 11312]

The author proves the following theorem which follows from work of Gauss and Hermite. If $M(f)$ is the minimum of a positive quadratic form in n variables with real coefficients and determinant Δ and $M(f)\leqq\gamma_n\Delta^{1/n}$, where γ_n is independent of the coefficients of f, then $\gamma_n\leqq\gamma_{n-1}^{(n-1)/(n-2)}$.

B. W. Jones (Ithaca, N. Y.).

Richmond, H. W. Notes on a problem of the "Waring" type. J. London Math. Soc. **19**, 38–41 (1944). [MF 11319]

The author describes calculations which have the following results: of the multiples of 5 from 0 to 20,000, more than 70% are the sum of three pyramidal numbers and all the others are the sum of four. This implies that every number less than 20,000 is the sum of six pyramidal numbers. The author found no cases in which six are required. From these and other considerations the author surmises that Dickson's result (that all sufficiently large numbers can be expressed as the sum of eight pyramidal numbers) is not the best possible one. *B. W. Jones* (Ithaca, N. Y.).

Pillai, S. S. On Waring's problem with powers of primes. J. Indian Math. Soc. (N.S.) **8**, 18–20 (1944). [MF 11273]

An improvement of some of the author's previous results [Proc. Indian Acad. Sci., Sect. A. **9**, 29–34 (1939); **12**, 202–204 (1940); these Rev. **2**, 35]. *H. S. Zuckerman.*

Selberg, Sigmund. Über die Verteilung einiger Klassen quadratfreier Zahlen, die aus einer gegebenen Anzahl von Primfaktoren zusammengesetzt sind. Skr. Norske Vid. Akad. Oslo. I. no. **5**, 49 pp. (1942). [MF 11187]

Let $\pi_n(x)$ be the number of integers not greater than x which are a product of exactly n different primes; it is known [E. Landau, Bull. Soc. Math. France **28**, 25–38 (1900)] that

$$\pi_n(x)=\frac{x}{\log x}\frac{(\log\log x)^{n-1}}{(n-1)!}+O\left(\frac{x}{\log x}(\log\log x)^{n-2}\right).$$

The author obtains the better result:

$$\frac{\log x}{x}\pi_n(x)=\sum_{\nu=1}^{n}\frac{a_\nu}{(n-\nu)!}(\log\log x)^{n-\nu}+O((\log\log x)^{-q});$$

the numbers a_1, a_2, $\cdots$ are absolute constants, and q is arbitrarily large; this is deduced as an elementary consequence of the prime number theorem. If m is any product of n different primes, let P_m denote the largest prime factor of m. The author proves that the number of $m<x$ satisfying the condition

$$P_m<x^{1-(\log x)^{-\phi(x)}}$$

is $o(x)$, where n is given and $\phi(x)$ is any positive function such that $\phi(x)\to 0$, $\phi(x)\log\log x\to\infty$, as $x\to\infty$.

C. L. Siegel (Princeton, N. J.).

Selberg, Atle. On the zeros of Riemann's zeta-function. Skr. Norske Vid. Akad. Oslo. I. no. **10**, 59 pp. (1942). [MF 11188]

Let $N_0(T)$ be the number of zeros of $\zeta(\sigma+it)$ with $\sigma=\frac{1}{2}$, $0<t<T$; $N(\sigma', T)$ the number with $\sigma>\sigma'$, $0<t<T$. If $a>\frac{1}{2}$, $U\geqq T^a$, the principal results are (I) $N_0(T+U)-N_0(T)>AU\log T$ for $T>T_0$ and (II) $N(\sigma, T+U)-N(\sigma, T)=O(U/(\sigma-\frac{1}{2}))$ uniformly for $\frac{1}{2}<\sigma\leqq 1$. These represent considerable improvements over known results and consequently yield improvements in certain corollaries. Thus from (I) there follows $N_0(T)>AT\log T$ for $T>T_0$ and that, if $\Phi(t)$ is a positive function, increasing to infinity, then for almost all $t>0$ there is a zero of $\zeta(\frac{1}{2}+it)$ between t and $t+\Phi(t)/\log t$. Analogous corollaries are deduced from (II). The lower bound for $N_0(T)$ combined with a well-known upper bound shows that $N_0(T)$ is of order $T\log T$. The method of proof depends on the usual arguments applied not to $\zeta(s)$ but to $\zeta(s)$ multiplied by a function that tends to smooth out the peculiarities of $\zeta(s)$.

H. S. Zuckerman (Seattle, Wash.).

Linnik, U. On the connexion of extended Riemann hypothesis with I. M. Vinogradow's method in the theory of primes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 145–146 (1943). [MF 11059]

The author announces a theorem on Dirichlet's L-series, which gives an upper bound for the number of those L-series (mod D) possessing zeros in a rectangle $1-\theta\leqq\sigma\leqq 1$, $|t|<D^{1/2}$. He mentions some deductions from this in the theory of primes. The paper is too condensed to be completely intelligible.

H. Davenport (Bangor).

Ramanathan, K. G. Multiplicative arithmetic functions. J. Indian Math. Soc. (N.S.) **7**, 111–116 (1943). [MF 10926]

Perhaps the most conspicuous class of numerical functions is that of multiplicative functions $f(n)$ for which

$$f(mn)=f(m)f(n), \qquad (m, n)=1, \tag{1}$$

holds for all pairs of coprime integers n and m. All solutions of (1) are obtained by assigning arbitrary values to $f(p^\alpha)$, where α ranges over integers and p over primes. In case $f(p^\alpha)$, as a function of α, is (for each prime p) a recurring series of the second order the function $f(n)$ is called a quadratic multiplicative function. An interesting example of such a function is Ramanujan's function $\tau(n)$ generated by

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}=\sum_{n=1}^{\infty}\tau(n)x^n.$$

In fact $\tau(p^{\alpha+1})=\tau(p)\tau(p^\alpha)-p^{11}\tau(p^{\alpha-1})$, a recurring series of the second order.

The object of this note is to obtain for such functions a generalization of (1) valid for any pair of integers m, n, coprime or not. The result is

$$f(mn)=\sum f(m/\delta)f(n/\delta)\mu(\delta)B(\delta), \tag{2}$$

where δ ranges over the common divisors of m and n; the function μ is the well-known Möbius function. Since this function vanishes whenever δ has a square factor greater than 1, we need define $B(k)$ for square-free numbers only. For such values of k, $B(k)$ is multiplicative and $B(p)=f^2(p)-f(p^2)$. For $f=\tau$, $B(p)=p^{11}$. Hence, for every pair n, m,

$$\tau(mn)=\sum_{\delta\mid(m,n)}\tau(m/\delta)\tau(n/\delta)\mu(\delta)\delta^{11}.$$

An identity similar to (2) is also discussed in case f is a so-called totient. For example, for Euler's totient function,

$$\varphi(m)\varphi(n)=\sum_{\delta\mid(m,n)}\varphi(mn/\delta)\mu(\delta).$$

D. H. Lehmer (Berkeley, Calif.).

Basoco, M. A. On certain arithmetical functions due to G. Humbert. Bull. Amer. Math. Soc. **50**, 547–555 (1944). [MF 10847]

The functions in question are integral functions of z and satisfy functional equations of the form

$$h(z+\pi)=(-)^a h(z), \quad h(z+\pi\tau)=(-)^b h(z)+F_{ab}^{(n)}(z),$$

where a, b are 0 or 1 and $F_{ab}^{(n)}(z)$ is an expression which involves $\vartheta_\alpha(z)$. The solution $H_{ab}^{(n)}(z)$ of the functional equations is uniquely determined except when $a=b=0$, when the additional condition $H_{00}^{(n)}(0)=0$ makes the determination unique. The solutions are given in the form of Fourier series; for instance,

$$H_{00}^{(2)}(z)=4\sum_{n=1}^{\infty}(q^{n(n+1)}/(1-q^{2n}))\sin 2nz,$$

where $q=e^{i\pi\tau}$. The Fourier series can be converted into an "arithmetized form," for example,

$$H_{00}^{(2)}(z)=4\sum_{n=1}^{\infty}q^n\{\sum'\sin 2\delta z\},$$

where $\sum'$ refers to the summation over the conjugate divisors (d, δ) of n such that $\delta<d$ and $\delta-d\equiv 1$ (mod 2). The comparison of the arithmetized series with the trigonometric developments for certain theta quotients leads to identities between $H_{ab}^{(n)}(z)$ and theta functions. These identities may be paraphrased into arithmetical relations of which a sample is

$$\sum f(x_1)=2\sum(t+\tau-4k)f(\tau+2k),$$

where $f(x)$ is an arbitrary even function, β is a positive integer of the form $4k+3$ partitioned as follows:

$$\beta=x_1^2+x_2^2+x_3^2+w_1^2+w_2^2=4k^2+tz, \qquad \tau<t,$$

and the summation extends over all partitions. The paraphrased formulae with $f(x)=1$ yield enumerations relative to the number of representations of a number as a sum of five squares. There are also applications to relations between the greatest integer function and certain incomplete numerical functions of the divisors of an integer.

A. Erdélyi (Edinburgh).

Chabauty, Claude. Sur le théorème fondamental de la théorie des points entiers et pseudo-entiers des courbes algébriques. C. R. Acad. Sci. Paris **217**, 336–338 (1943). [MF 11101]

C. L. Siegel has proved [Abh. Preuss. Akad. Wiss. **1929**, no. 1, 70 pp.] that an algebraic curve $f(x, y)=0$ of positive genus contains only a finite number of points with integral coordinates x, y in any given algebraic number field K; the proof uses the theory of Abelian functions and a proposition concerning simultaneous approximation of algebraic numbers. The author indicates the idea of a much simpler proof. He asserts that there exist on the curve two integral algebraic functions ξ and η, ramified only at infinity and cyclically, and satisfying an equation $\phi(\xi, \eta)=\text{const.}$, where ϕ is a binary form with at least three distinct linear factors; in virtue of A. Weil's principle of transport, the

general Diophantine equation $f=0$ is then reduced to the particular type $\phi=$const. which had been previously discussed by A. Thue [Skrifter Vid. Selsk. Christiania **1908**, no. 3] and C. Siegel [Math. Z. **10**, 173–213 (1921)]. This proof can be extended to the case that x, y are fractional numbers in K provided all prime ideal factors of the ideal $(x, y, 1)^{-1}$ belong to a preassigned finite set.

C. L. Siegel (Princeton, N. J.).

ANALYSIS

Theory of Functions of Complex Variables

***Nevanlinna, Rolf. Eindeutige analytische Funktionen.** J. W. Edwards, Ann Arbor, Michigan, 1944. viii+353 pp. $9.65.

Reprint of volume 46 of the collection: Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, published by J. Springer, Berlin, 1936.

Meymann, N. On the problem of Hermite-Hurwitz for integer transcendental functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 46–49 (1943). [MF 11173]

Let the integral function $F(z)$ be written as $F(z)=g(z)+ih(z)$, where $g(z)$ and $h(z)$ are integral and have real coefficients (expanded about $z=0$), and suppose they have no zeros in common. Classes H, B, $\bar{B}$ of functions $F(z)$ are defined: $F(z)$ belongs (i) to class H if it has no zeros in the lower half-plane; (ii) to class B if, on writing $\bar{F}(z)=g(z)-ih(z)$, then, to each $\epsilon>0$, $|F(z)/\bar{F}(z)|<e^{\epsilon|z|}$ for all sufficiently large z lying in the upper half-plane; (iii) to class $\bar{B}$ if it belongs to both H and B.

Theorems concerning these classes are obtained. Among them are the following. (i) If $F(z)\epsilon\bar{B}$, then $g(z)$ and $h(z)$ form a real couple (that is, g and h have real coefficients, have no common zero, and for every pair of real numbers λ, μ the function $\lambda g+\mu h$ has no imaginary zeros), and $I\{g(z)/h(z)\}\div I\{z\}<0$ ($I\{u\}\equiv$imaginary part of u). Conversely, if this I-inequality holds, then g, h form a real couple and $F(z)\epsilon\bar{B}$. (ii) $F(z)\epsilon\bar{B}$ if and only if the following conditions hold: (a) the zeros of g and h are real and interlacing, and (b) if a_0 is the absolutely least zero of $g(z)$, and b_0 the smallest zero of $h(z)$ exceeding a_0, then there exist constants A, B having the same sign, and real integral functions g_1, h_1 such that

$$g(z)=Ae^{g_1(z)}(z-a_0)\prod_1^\infty (1-z/a_n)e^{p_n(z/a_n)},$$

$$h(z)=Be^{h_1(z)}(z-b_0)\prod_1^\infty (1-z/b_n)e^{p_n(z/b_n)}$$

and

$$g_1(z)-h_1(z)+\sum_{n=1}^\infty [p_n(z/a_n)-p_n(z/b_n)]\equiv 0.$$

Here the p_n's are Weierstrass exponents. *I. M. Sheffer.*

Meyman, N. On the distribution of the zeroes of an integer function. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 179–181 (1943). [MF 11162]

Let

$$\zeta(z)=\frac{F(z)}{\bar{F}(z)}=\frac{g(z)+ih(z)}{g(z)-ih(z)}$$

[see the preceding review]. For each real number a (finite or infinite), roots of the equation $h(z)/g(z)=a$ that lie in the closed upper half-plane ($y\geqq 0$, where $z=x+iy$) are upper a-points. If an upper a-point z is a zero of order k, then its upper multiplicity is defined to be (i) k if $I\{z\}>0$; (ii) $k/2$ if z is real and k is even; and (iii) $(k-1)/2$ or $(k+1)/2$ if z is real and k is odd, according as $\arg F(z)$ increases or decreases at point z. For a given a, the sum of all upper multiplicities is denoted by $S_a(h/g)$. Now $F(z)$ is said to belong to class B if for every $\eta>0$ the inequality $|\zeta(z)|<1+\eta$ holds for all sufficiently large z lying in the upper half-plane. [This is equivalent to condition B of the preceding review.]

In terms of the above definitions, the following are some of the results obtained. (i) If $F(z)\epsilon B$, then, for all a different from the (unique) asymptotic value of $\zeta(z)$ in the upper half-plane, the numbers $S_a(h/g)$ have a finite common value, which value exceeds the quantity $S_b(h/g)$, where b is the asymptotic value. (ii) If two numbers a and b exist (one of which may be infinite) such that $S_a(h/g)$, $S_b(h/g)$ are finite, then either (a) $F(z)\epsilon B$ and the number of zeros of $F(z)$ in the lower half-plane is equal to the greater of $S_a(h/g)$, $S_b(h/g)$, or (b) $F(z)$ has infinitely many zeros in the lower half-plane, and $F(z)\bar{\epsilon}B$. Erratum: in line 1 (page 181) read "lower" instead of "upper." *I. M. Sheffer.*

Delange, Hubert. Sur les suites de fonctions entières à zéros réels et négatifs. C. R. Acad. Sci. Paris **217**, 225–227 (1943). [MF 10646]

Let $\{f_n(z)\}$ be a sequence of entire functions of genus p with real and negative zeros. Let $f_n(0)=0$. We denote by $\nu_n(x)$ the number of zeros of $f_n(z)$ not less than $-x$, $x>0$, and by $\varphi(n)$ a positive function of n. Under the condition that all the zeros are less than a fixed negative number $-a$, the author investigates the convergence of

$$\{\varphi(n)\}^{-1}\log f_n(z)$$

in the plane cut along the real segment $x\leqq -a$. One of his conditions is that $\{\varphi(n)\}^{-1}\nu_n(x)$ converges. Similar questions are discussed under the condition that the zeros are real and negative without any upper bound $-a$. Based on these results a new proof is given for an older theorem of Valiron [Ann. Univ. Toulouse (3) **5**, 117–257 (1913), in particular, p. 230]. *G. Szegö* (Stanford University, Calif.).

Levin, B. Hermite's criterion for integral functions of exponential type. I. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 47–50 (1943). [MF 11051]

Hermite showed that all the roots of the polynomial $f(x)=P(x)+iQ(x)$, where P and Q are real, lie in the upper half-plane if and only if all the roots of $P(x)$ and $Q(x)$ are real and alternate and, for some real x, $Q'(x)P(x)-P'(x)Q(x)>0$. This criterion has been extended to entire functions of genus 0 and in special cases to functions of exponential type. Here the author shows that the entire function of exponential type $f(x)=P(x)+iQ(x)$, where P and Q are real, satisfies conditions (A) and (B) if and only if it satisfies conditions (C) and (D). (A) All the roots of $f(z)$ lie in the upper half-plane. (B) The type $h(\pi/2)$ of $f(z)$ on the ray $\arg z=\pi/2$ is no greater than $h(-\pi/2)$. (C) All roots a_k of $P(x)$ and b_k of $Q(x)$ are real and satisfy $a_k\leqq b_k\leqq a_{k+1}$. (D) Writing

$$P(x)=Ae^{\alpha x}\prod(1-x/a_k)e^{x/a_k},$$

and a similar formula for $Q(x)$, then

$$a-b+\sum(1/a_k-1/b_k)=0.$$

N. Levinson (Cambridge, Mass.).

Levin, B. Hermite's criterion for integral functions of exponential type. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 99–100 (1943). [MF 11055]

Here the author shows that the conditions (B) and (D) of his theorem cited in the above review are essential by proving that conditions (A) and (C) imply the conditions (B) and (D). Let

$$f(z)=\sum_{k=1}^{n} e^{\Omega_k z}P_k(z), \qquad \lambda_1<\lambda_2<\cdots<\lambda_n,$$

where $P_k(z)$ are polynomials and $\lambda_n>-\lambda_1$. Then the author shows that the original criterion of Hermite for the roots of a polynomial cited above applies to $f(z)$. *N. Levinson.*

Boas, R. P., Jr. Functions of exponential type. III. Duke Math. J. 11, 507–511 (1944). [MF 11120]

[Cf. Duke Math. J. 11, 9–15, 17–22 (1944); these Rev. 5, 175.] Let $C(c)$ denote the class of entire functions $f(z)$ of exponential type c with

$$f(z)=\sum_{k=0}^{\infty} a_k z^k, \quad s_n(z)=\sum_{k=0}^{n} a_k z^k.$$

The author obtains the following theorem. Let L be a distributive operator carrying each element of $C(c)$ into an element of $C(c)$ and such that, for $f(z)\in C(c)$,

$$L[f']=\{L[f]\}', \quad \lim L[s_n(z)]=L[f(z)].$$

Then L has the form

$$\lambda(d/dz)=\sum_{n=0}^{\infty}\lambda_n d^n/dz^n \tag{1}$$

and $\lambda(z)$ is analytic in $|z|\leqq c$. For more general classes the author has a similar characterization of operators. Let C be a distributive class of functions analytic at $z=0$, containing all polynomials, closed under differentiation, and having a topology in which partial sums of the power series of a function of the class converge to the function. Let L be a distributive operator, permutable with differentiation, and continuous in the topology of C. Then there is a sequence $\{\lambda_n\}_{n=0}^{\infty}$ such that

$$L[f(z)]=\sum_{n=0}^{\infty} a_n n!\sum_{k=0}^{n}\lambda_k z^{n-k}/(n-k)!,$$

where $f(z)=\sum_0^\infty a_n z^n$, $f(z)\in C$. Whenever the class C permits the order of summation above to be changed, then

$$L[f(z)]=\sum_{k=0}^{\infty}\lambda_k f^{(k)}(z),$$

which is of the form (1). Finally, the author obtains a topology for $C(c)$ in which every operator of form (1) is continuous, and in which $s_n\to f$. *M. S. Robertson.*

Nachbin, Leopoldo. An extension of the notion of integral functions of the finite exponential type. Anais Acad. Brasil. Ci. 16, 143–147 (1944). [MF 10966]

Let $\{\alpha_n\}$ be a sequence of real numbers $0<\alpha_0\leqq\alpha_1\leqq\cdots\leqq\alpha_n\leqq\cdots$, $\alpha_n\to+\infty$. Define the integral function $e(z)$ as

$$\sum_{0}^{\infty} z^n/(\alpha_0\alpha_1\cdots\alpha_n),$$

a generalization of the exponential function e^z. Let $f(z)$ be an integral function. Then $f(z)$ is said to be of the finite $\{e(z);s\}$ type if s is the smallest nonnegative real number having the property that to each $\epsilon>0$ there is a $K(\epsilon)>0$ so that, for each complex z,

$$|f(z)|\leqq K(\epsilon)e[(s+\epsilon)|z|].$$

With this extension of the notion of integral function of finite exponential type, the author establishes the following extension of a well-known theorem. A necessary and sufficient condition that the integral function $f(z)$ should be of the finite $\{e(z);s\}$ type is that

$$\limsup_{n\to+\infty}|(\alpha_0\alpha_1\cdots\alpha_n)a_n|^{1/n}=s,$$

where $f(z)=\sum_0^\infty a_n z^n$. Some examples of integral functions $e(z)$ are given. *M. S. Robertson* (New Brunswick, N. J.).

Hervé, M. Sur quelques applications de la notion d'ordre précisé. Bull. Sci. Math. (2) 66, 17–24, 31–48 (1942). [MF 10462]

For a given point set E of the plane, an entire function satisfying certain conditions relative to the order, to the type or to the indicatrix of growth, and which vanishes on E, is identically zero. Gelfond has obtained results for the set of points $n^{1/\rho}$, n a positive integer, ρ the order of the function. Another attack has been made by Ganapathy Iyer for the set E constructed from the sequence $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots\to\infty$ where, $n(r)$ being the number of $\lambda_n\leqq r$,

$$0<A=\varliminf_{r\to+\infty}\frac{n(r)}{r^\rho}\leqq\varlimsup_{r\to+\infty}\frac{n(r)}{r^\rho}=B<+\infty. \tag{1}$$

The author extends these methods to more general sets E obtained by replacing ρ by a precise order $\rho(x)$ satisfying these conditions: $\rho(x)$ is defined and continuous for $x\geqq 0$, $\rho(0)>0$; it has a derivative save at isolated points where its left and right derivatives exist;

$$0<\beta=\varliminf_{x\to+\infty}\rho(x)\leqq\varlimsup_{x\to+\infty}\rho(x)=\rho<+\infty,$$

$x\rho'(x)\log x\to 0$ as $x\to+\infty$ and $x^{\rho(x)}$ increases with x. If $\rho=\beta$, $\rho(x)$ is a Lindelöf order L.

The set $n^{1/\rho}$ of Gelfond may be considered as values of the inverse function x of $y=x^\rho$ for integers y. The author replaces the set $n^{1/\rho}$ by numbers $\alpha_n=\sigma(n)$ which are values for integral y of the function $x=\sigma(y)$ which is the inverse of the function $y=x^{\rho(x)}$. In extending Ganapathy Iyer's results the author replaces ρ in (1) by $\rho(r)$ for the hypothesis on the sequence $\{\lambda_n\}$. The indicatrix of the growth of the entire function $f(z)$ relative to the order $\rho(x)$ is defined as the function

$$h(\theta)=\varlimsup_{r\to+\infty}\frac{\log|f(re^{i\theta})|}{r^{\rho(r)}}.$$

Several theorems are obtained relative to $h(\theta)$ which are then used to determine the author's principal results. These include the following.

If $\rho(x)$ is any order with $\beta>\frac{1}{2}$, $\rho<4\beta$, if $f(z)$ is holomorphic and of order not greater than ρ for $|\arg z|<\gamma$, $(\pi/2\beta)<\gamma$, if the indicatrix $h(\theta)$ relative to $\rho(x)$ omits $+\infty$ in the neighborhood of $\theta=\pm\pi/2\rho$ and $\theta=\pm\pi/2\beta$, and for $\pi/2\rho\leqq|\theta|\leqq\pi/2\beta$ satisfies

$$h(\theta)<\pi\varlimsup_{r\to+\infty}\frac{\cos[(|\theta|-\pi/p)\rho(r)]}{\sin(\pi\rho(r)/p)}=H_p(\theta),$$

where p is a convenient fixed value, $\rho<p\leqq 4\beta$, and if $f(a_n)=0$ for all n, then $f(z)\equiv 0$. The theorem holds again for $\beta\leqq\frac{1}{2}$ when $f(z)$ is an entire function of order not greater

than ρ. If $\rho(x)$ is any order, $\rho<\frac{1}{2}$, $f(z)$ an integral function of order not greater than ρ, if $h(\pi)<\pi$ cosec $\pi\rho$, and if $f(a_n)=0$ for all n, then $f(z)\equiv 0$. If $\frac{1}{2}<\rho<1$, and $h(\theta)\not\equiv+\infty$ in the neighborhood of $\theta=\pm\pi/2\rho$ with $h(\pm\pi/2\rho)<H_1(\pm\pi/2\rho)$, and if $f(a_n)=0$ for all n, then $f(z)\equiv 0$.

Two theorems dealing with the extension of the method of Ganapathy Iyer are also given. Finally, a comparison of the results of the two methods is made.

M. S. Robertson (New Brunswick, N. J.).

Dvorkine, B. Sur le développement d'une fonction entière d'une variable complexe en série convergente de Newton. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 377–380 (1943). (Russian. French summary) [MF 10231]

The entire function $f(z)$ can be developed in a convergent Newton interpolation series based on the points $(\log n)^s$, $n=1, 2, \cdots$, provided that

$$|f(z)|<\exp\{(1-\epsilon)\log(\alpha-1)\exp([|z|/\alpha]^{1/s})\},$$

with $\alpha>2$, $0<\epsilon<1$. *R. P. Boas, Jr.*

McMillan, Audrey Wishard. A Phragmén Lindelöf theorem. Amer. J. Math. **66**, 405–410 (1944). [MF 10910]

An extension of the Phragmén-Lindelöf theorem due to F. Wolf [J. London Math. Soc. **14**, 208–216 (1939); these Rev. **1**, 48] is improved in the present paper. It is shown that, if $f(z)$ is analytic in $\Re(z)\geqq 0$, if $|f(iy)|\leqq 1$, and if there is a Lebesgue integrable function $\psi(\theta)$ in $(-\pi/2, \pi/2)$ such that

$$|f(re^{i\theta})|\leqq\exp[r\psi(\theta)],$$

then there is a number k, not necessarily positive, satisfying

$$k\leqq\varliminf_{r\to\infty}\frac{\log|f(re^{i\theta})|}{r\cos\theta}$$

for any θ in $(-\pi/2, \pi/2)$, such that $|f(x+iy)|\leqq e^{kx}$. In the result of F. Wolf, under the assumption that for every positive ϵ there is an $R(\epsilon)$ such that

$$|f(re^{i\theta})|\leqq\exp[\epsilon r\psi(\theta)]$$

for $r>R(\epsilon)$, it is concluded that $|f(z)|<1$.

E. F. Beckenbach (Austin, Tex.).

Wolff, J. La représentation conforme au voisinage d'un point frontière. Nederl. Akad. Wetensch., Proc. **45**, 169–170 (1942). [MF 10384]

A theorem of J. Ferrand [C. R. Acad. Sci. Paris **212**, 977–980 (1941); these Rev. **5**, 37] is generalized as follows. Let α denote an accessible boundary point of a simply-connected region Δ of the ζ-plane and let $\zeta=\zeta(z)$ map the right-half z-plane (1, 1) and directly conformally onto Δ in such a manner that $z=0$ corresponds to the prime end containing α. Then α is the angular limit of $\zeta(z)$ as z tends to zero. The proof is based upon the use of Schwarz's inequality for obtaining inequalities involving area and length.

M. H. Heins (Washington, D. C.).

Martinelli, Enzo. Sopra una dimostrazione di R. Fueter per un teorema di Hartogs. Comment. Math. Helv. **15**, 340–349 (1943).

Der bekannte Hartogs'sche Satz, dass eine Funktion $f(z_1, z_2, \cdots, z_n)$, $n\geqq 2$, die in sämtlichen Randpunkten eines schlichten beschränkten Bereiches mit zusammenhängendem Rande regulär ist, sich in jeden inneren Punkt des Bereiches hinein analytisch fortsetzen lässt, wurde von R. Fueter mit Hilfe der Theorie der rechtsregulären Funktionen einer Quaternionenvariablen beziehungsweise einer allgemeineren hyperkomplexen Variablen neu bewiesen [Comment. Math. Helv. **12**, 75–80 (1939); **14**, 394–400 (1942); these Rev. **1**, 115; **4**, 139]. In der vorliegenden Arbeit wird ein Beweis des Hartogs'schen Satzes gegeben, der sich an den Beweis Fueters anlehnt, aber innerhalb der Theorie der Funktionen mehrerer komplexer Veränderlichen geführt wird und von einer vom gleichen Verfasser bewiesenen Integralformel ausgeht. Ferner gibt der Verfasser für den Sonderfall konvexer Bereiche einen Beweis an, der in Evidenz setzt, weshalb aus topologischen Gründen der Hartogs'sche Satz für $n\geqq 2$, nicht aber für $n=1$ gilt. *P. Thullen* (Quito).

Severi, Francesco. A proposito d'un teorema di Hartogs. Comment. Math. Helv. **15**, 350–352 (1943).

Der Verfasser zeigt, wie sein früher bewiesener Satz, daß eine Funktion einer reellen und einer komplexen Veränderlichen, die auf der geschlossenen zusammenhängenden Randfläche eines Bereiches des Raumes R_3 analytisch ist, notwendig ins ganze Innere des Bereiches hinein analytisch fortsetzbar ist, sich ohne Schwierigkeit auf Funktionen von $n-1$ reellen und einer komplexen Veränderlichen (oder allgemeiner: q reellen und $n-q$ komplexen Veränderlichen) ausdehnen läßt. Durch den Übergang vom Reellen zum Komplexen gelangt man dann auf einfachstem Wege zum Hartogs'schen Satz für n komplexe Veränderliche, der besagt, daß eine Funktion $f(z_1, z_2, \cdots, z_n)$ von n komplexen Veränderlichen, die sich in allen Punkten einer geschlossenen zusammenhängenden Randhyperfläche eines Bereiches des R_{2n} analytisch verhält, sich notwendig ins ganze Innere des Bereiches hinein fortsetzen läßt. Derselbe Satz kann auch aus dem entsprechenden Satz für $n=2$ abgeleitet werden, wenn man die Gültigkeit des ebenfalls von Hartogs bewiesenen Satzes voraussetzt, daß eine Funktion $f(z_1, z_2, \cdots, z_n)$, die in einem Bereiche nach jeder einzelnen Veränderlichen analytisch ist, dort schlechthin analytisch ist. *P. Thullen.*

Polynomials, Polynomial Approximations

Lax, Peter D. Proof of a conjecture of P. Erdös on the derivative of a polynomial. Bull. Amer. Math. Soc. **50**, 509–513 (1944). [MF 10841]

If $p_n(z)$ is a polynomial of degree n and $|p_n(z)|\leqq 1$ for $|z|\leqq 1$, then Bernstein's well-known theorem asserts that $|p_n'(z)|\leqq n$ for $|z|\leqq 1$. Erdös conjectured that, if $p_n(z)$ has no zeros inside the unit circle, then its derivative satisfies the stronger inequality $|p_n'(z)|\leqq\frac{1}{2}n$ for $|z|\leqq 1$. The author gives a proof of this conjecture. He also shows that, if $p_n(z)$ has no zeros inside the unit circle, then

$$\left\{\int_{-\pi}^{\pi}|p_n'(e^{i\theta})|^2d\theta\right\}^{\frac{1}{2}}\leqq(n/\sqrt{2})\left\{\int_{-\pi}^{\pi}|p_n(e^{i\theta})|^2d\theta\right\}^{\frac{1}{2}}.$$

This is a sharper form of Zygmund's inequality in which the factor $n/\sqrt{2}$ on the right hand side is replaced by n [Proc. London Math. Soc. (2) **34**, 392–400 (1932)].

A. C. Offord (Newcastle-on-Tyne).

Bellman, Richard. A note on an inequality of E. Schmidt. Bull. Amer. Math. Soc. **50**, 734–736 (1944). [MF 11286]

By use of Legendre polynomials the author gives an elementary proof of Schmidt's theorem to the effect that a

polynomial $f(x)$ of degree N satisfies an inequality

$$\left[\int_{-1}^{1} f'^2(x)dx\right]^{\frac{1}{2}} \leqq k_N N^2 \left[\int_{-1}^{1} f^2(x)dx\right]^{\frac{1}{2}},$$

where k_N depends on N alone. More general results are known [see the papers cited by Bellman]; the purpose here is to give a simple proof in the above special case. The chain of inequalities on page 735 is somewhat mystifying and, the reviewer believes, incorrect. The argument can be rectified, however, and the use of Schwarz' inequality at an earlier stage yields a better bound for k_N. The recursion formula $P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x)$ is misprinted on page 734. *H. Pollard* (New York, N. Y.).

Delange, Hubert. Sur certaines suites de polynomes. C. R. Acad. Sci. Paris **217**, 191–193 (1943). [MF 10642]

Let $P_n(z)$ be a sequence of polynomials with the zeros $\alpha_{n\rho}$; let $\varphi(n)$ be a positive function of n and $\nu_n(e)$ the number of zeros $\alpha_{n\rho}$ belonging to a given set e. In his thèse [Ann. Sci. École Norm. Sup. (3) **56**, 173–275 (1939); these Rev. **1**, 310] the author studied the sequence

$$(*) \qquad \{\varphi(n)\}^{-1} \log |c_n P_n(z)|,$$

where the sequence $\{c_n\}$ is chosen in a proper way. In the special case when all the zeros of $P_n(z)$ are real, more precise results can be obtained which are announced in the present note. In order that a sequence $\{c_n\}$ exists such that (*) is convergent outside of the real axis, the following conditions are necessary and sufficient. The sequence $\{\varphi(n)\}^{-1}\nu_n(e)$ tends to a limit $\nu(e)$, and the expressions

$$\{\varphi(n)\}^{-1} \sum_{\rho} \frac{\alpha_{n\rho}}{1+\alpha_{n\rho}^2}, \qquad \{\varphi(n)\}^{-1} \sum_{\rho} \frac{1}{1+\alpha_{n\rho}^2}$$

have limits as $n\to\infty$. Then $c_n=P_n(z_0)$ can be taken, where z_0 is arbitrary outside the real axis. The limit function $U(z)$ of (*) is studied. Extension of these results is possible to the case when the zeros are real and bounded from one side. Laguerre polynomials furnish an example for this case. *G. Szegö* (Stanford University, Calif.).

Gavrilov, L. On K-prolongeable polynomials. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 246–249 (1942). [MF 8664]

With AB an arbitrary arc on the unit circle, K_1 a dense sequence on the remainder, and $K=AB+K_1$, the author proves that every polynomial is K-prolongable. This means it is always possible to add terms of higher order than the given ones so that the resulting polynomial has all its roots in K. *P. Erdös* (Ann Arbor, Mich.).

Jackson, Dunham. Boundedness of orthonormal polynomials on loci of the second degree. Duke Math. J. **11**, 351–365 (1944). [MF 10673]

In various papers the author has studied polynomials in x and y which are orthogonal and normalized on a given part of a given algebraic curve (with the arc element in the integral), admitting also a certain positive weight function. The question arises whether weight functions exist for which the complete set of these polynomials is uniformly bounded on the whole set of integration, or on a certain part of it. (If such a weight function exists then infinitely many others follow by using a theorem due to Korous in the case of a line segment and extended by the author to this more general situation.) The author investigates the existence of such weight functions for arcs or suitable systems of arcs on curves of the second degree. *G. Szegö* (Stanford University, Calif.).

Basu, K. A note on Sonine's polynomials. II. Bull. Calcutta Math. Soc. **35**, 127–130 (1943). [MF 11009]

Continuation of the study of various elementary properties of Sonine's polynomials. The first part appeared in the same Bull. **35**, 21–32 (1943); these Rev. **5**, 180. *G. Szegö* (Stanford University, Calif.).

Marcouchevitch, A. Sur les polynomes de Faber. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 49–60 (1944). (Russian. French summary) [MF 11118]

The Faber polynomials as originally defined provide an expansion for a function analytic in the closed finite region bounded by an analytic curve. The case of a region bounded by a rectifiable curve has been discussed by Heuser [Deutsche Math. **4**, 451–454 (1939); these Rev. **1**, 14]. The author considers a simply connected region G, containing the point at infinity, whose complement consists of a finite number of closed simply connected regions g_k bounded by rectifiable Jordan curves and of a finite number of rectifiable Jordan arcs. By using a function which maps G conformally on the exterior of the unit circle, the author constructs polynomials in terms of which there is an expansion (in general not unique) for every function analytic in the interior of each g_k and of bounded absolute value (or even satisfying a more general condition). The series is uniformly summable (C, δ), $\delta>0$, in every closed subset interior to the g_k, and uniformly convergent in such a subset if the function has boundary values of bounded variation. For the case of only one region g, there is convergence under the more general condition that $|f(z)|\log^+|f(z)|$ is integrable on the boundary γ of g, provided that γ has a slope angle which, as a function of the arc length, satisfies a Hölder condition. *R. P. Boas, Jr.*

Touretzky, A. Sur la problème de la détermination de la borne d'un polynôme interpolatoire. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 194–196 (1943). [MF 11065]

Suppose $P_{2n+1}(x)$ is a polynomial of degree $2n+1$, defined in $-1\leqq x\leqq 1$, whose modulus does not exceed unity at $2n+2$ equidistant points of the intervals $(-1, -\alpha)$ and $(\alpha, 1)$ for $0<\alpha<1$. The author determines an upper bound for the modulus of $P_{2n+1}(x)$ in the interval $(-1, 1)$. This is given as the greater of two rather complicated expressions. These expressions naturally depend on α and they have their least value when $\alpha=\frac{1}{2}$. Hence when $\alpha=\frac{1}{2}$ the upper bound of $|P_{2n+1}(x)|$ is required to be least and for large n it cannot exceed a quantity which is asymptotic to

$$\frac{(1+\sqrt{3})^{\frac{1}{2}}}{2\sqrt{2}}\left(\frac{3\sqrt{3}}{2}\right)^n.$$

A. C. Offord (Newcastle-on-Tyne).

Geronimus, J. On polynomials orthogonal on the circle, on trigonometric moment-problem and on allied Carathéodory and Schur functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 291–295 (1943). [MF 10734]

The paper announced without proof a number of results concerning the orthogonal polynomials $P_n(z)=h_n z^n+\cdots$ defined by the condition

$$(2\pi)^{-1}\int_{-\pi}^{\pi} P_n(z)\overline{P_m(z)}d\sigma(\theta)=\epsilon_{nm}, \qquad z=e^{i\theta}.$$

Here $\sigma(\theta)$ is a bounded nondecreasing function with infinitely many points of increase. Let

$$c_n=(2\pi)^{-1}\int_{-\pi}^{\pi}e^{-in\theta}d\sigma(\theta),$$

$|c_{i-k}|^2=D_n$, $|c_{i-k+1}|^2=\Delta_n$. The convergence of $\sum|\Delta_n/D_n|^2$ is necessary and sufficient for the convergence of $\sum|P_n(z)|^2$ for every $|z|<1$. Under this condition the asymptotic formula

$$h_n^{-1}P_n(z)\sim z^n p(1/z), \qquad |z|>1,$$

holds, where $p(z)$ is analytic for $|z|<1$. In case the series $\sum|\Delta_n/D_n|$ is convergent, the asymptotic formula

$$h_n^{-1}P_n(z)=z^n\overline{p(z)}+O\left(\sum_{\nu=n}^{\infty}|\Delta_\nu/D_\nu|\right)$$

holds and $\sigma(\theta)$ will be absolutely continuous. The formal relation of the polynomials $P_n(z)$ to a certain algorithm introduced by I. Schur [J. Reine Angew. Math. 147, 205–232 (1917)] is also studied. *G. Szegö.*

Geronimus, J. On Gauss' and Tchebycheff's quadrature formulas. Bull. Amer. Math. Soc. 50, 217–221 (1944). [MF 10200]

Gauss' quadrature formula

$$\int_{-\infty}^{\infty}G_k(x)d\psi(x)=\sum_{i=1}^{n}\rho_i^{(n)}G_k(\xi_i^{(n)})$$

holds for every polynomial $G_k(x)$ of degree $k\leqq 2n-1$, the $\{\xi_i^{(n)}\}$ being roots of the polynomial $P_n(x)$, orthogonal with respect to the distribution $d\psi(x)$ ($i=1, \cdots, n$; $n=1, \cdots$). If the $P_n(x)$ are the Tchebycheff polynomials, the $\rho_i^{(n)}$, $i=1, \cdots, n$, are equal. The author gives four proofs that for no other choice of $\psi(x)$ are the $\rho_i^{(n)}$ all equal. *P. Erdös* (Ann Arbor, Mich.).

Mikeladze, Š. E. Quadrature formulas using differences. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 1001–1003 (1942). (Russian. Georgian summary) [MF 10323]

Using Taylor's formula with the remainder in integral form, and applying Newton's interpolation formula, the author derives a new quadrature formula including derivatives and differences, which contains as special cases some formulas obtained before. As a special case of this formula the author cites

$$\int_{a-h}^{a+h}\phi(x)dx=2h\phi(a-h)+2h^2\phi'(a)+h^2\sum_{\lambda=1}^{r}\sigma_\lambda\Delta^\lambda\phi'(a-\lambda h)+R,$$

where

$$\sigma_\lambda=(1/\lambda!)\int_{-1}^{1}(1-t)t^{(-\lambda)}dt, \quad t^{(-\lambda)}=t(t+1)\cdots(t+\lambda-1),$$

$$R=(h^{r+2}/(r+1)!)\int_{-1}^{1}(1-t)t^{(-r-1)}\phi^{(r+2)}(a+\tau h)dt,$$

where τ is between $-r$ and 1. *J. D. Tamarkin.*

Greville, T. N. E. A generalization of Waring's formula. Ann. Math. Statistics 15, 218–219 (1944). [MF 10830]

The following generalization of Waring's (usually called Lagrange's) interpolation formula is here given. [Reviewer's formulation.] Let $f(x)$ be analytic in a region containing the points $a_0, a_1, \cdots, a_n$. Let

$$\phi(x)=(x-a_0)^{m_0+1}(x-a_1)^{m_1+1}\cdots(x-a_n)^{m_n+1}$$

and

$$\phi_k(x)=(m_k+1)!\phi(x)/\{(x-a_k)^{m_k+1}\phi^{(m_k+1)}(a_k)\}.$$

Finally, let $P_k(x-a_k)\phi_k(x)/\phi(x)$ be the principal part of the function $f(x)/\phi(x)$ in the pole a_k. Then, the polynomial

$$F(x)=\sum_{i=0}^{n}P_i(x-a_i)\phi_i(x)$$

has the property that, for $k=0, 1, \cdots, n$, $F(a_k)=f(a_k)$, $F'(a_k)=f'(a_k)$, $\cdots$, $F^{(m_k)}(a_k)=f^{(m_k)}(a_k)$. The theorem is verified by the author through differentiation of $F(x)$, but, as it seems to the reviewer, may also be proved from the definitions of P_k and F. For, in the neighborhood N of $x=a_k$, we may write

$$f(x)=P_k(x-a_k)\phi_k(x)+\phi(x)\psi_k(x)$$
$$=F(x)+(x-a_k)^{m_k+1}\Psi_k(x),$$

where $\psi_k(x)$ and $\Psi_k(x)$ are analytic in N. *M. Marden.*

Special Functions

Kesava Menon, P. Some inequalities involving the Γ- and ζ-functions. Math. Student 11, 10–12 (1943). [MF 10991]

Using integral representations of the respective functions, the author derives some elementary inequalities. *O. Szász* (Cincinnati, Ohio).

***Strutt, M. J. O. Lamésche, Mathieusche, und verwandte Funktionen in Physik und Technik.** J. W. Edwards, Ann Arbor, Michigan, 1944. viii+116 pp. $3.25.

Reprint of Heft 3 of vol. 1 of the collection: Ergebnisse der Mathematik und ihrer Grenzgebiete, published by J. Springer, Berlin, 1932.

Sharma, J. L. Integral equations involving Lamé functions of complex degrees. Proc. Benares Math. Soc. (N.S.) 4, 19–25 (1943). [MF 10344]

The author considers Lamé's differential equation

$$(p^2-b^2)(p^2-c^2)d^2y/dp^2+p(2p^2-b^2-c^2)dy/dp$$
$$+\{(b^2+c^2)\alpha+n(n+1)p^2\}y=0,$$

where n is a complex number of the form $-\frac{1}{2}+pi$. Hobson has shown [Proc. London Math. Soc. (1) 23, 231–240 (1892)] that for certain values of α this equation possesses four solutions which are periodic functions of ϕ, where $\cos\phi=(p^2-b^2)^{\frac{1}{2}}(c^2-b^2)^{-\frac{1}{2}}$. These solutions, here denoted by $E_p^r(\phi)$, $L_p^r(\phi)$, $M_p^r(\phi)$, $N_p^r(\phi)$, are infinite series of the form $\sum_0^\infty c_j f_j(\phi)$, where $f_j(\phi)$ is $\cos 2n\phi$, $\sin 2n\phi$, $\cos(2n+1)\phi$, $\sin(2n+1)\phi$ in the four cases, respectively, and the constants c_j tend to zero. The author obtains the following homogeneous integral equations satisfied by these functions:

$$E_p^r(\phi)=\lambda_{rp}^{E}\int_{-\pi}^{\pi}\phi K_p(\sigma)(1+k\cos 2\theta)^{-\frac{1}{2}}E_p^r(\theta)d\theta,$$

$$L_p^r(\phi)=\lambda_{rp}^{L}\int_{-\pi}^{\pi}K_p''(\sigma)\sin 2\theta\sin 2\phi(1+k\cos 2\theta)^{-\frac{1}{2}}L_p^r(\theta)d\theta,$$

$$M_p^r(\phi)=\lambda_{rp}^{M}\int_{-\pi}^{\pi}K_p'(\sigma)\cos\theta\cos\phi(1+k\cos 2\theta)^{-\frac{1}{2}}M_p^r(\theta)d\theta,$$

$$N_p^r(\phi)=\lambda_{rp}^{N}\int_{-\pi}^{\pi}K_p'(\sigma)\sin\theta\sin\phi(1+k\cos 2\theta)^{-\frac{1}{2}}N_p^r(\theta)d\theta.$$

Here K_p is Mehler's conal function of degree $-\frac{1}{2}+pi$, $k=(c^2-b^2)/(c^2+b^2)$, and

$$\sigma=\{(1+k\cos 2\theta)(1+k\cos 2\phi)/(1-k^2)\}^{\frac{1}{2}}.$$

The author obtains orthogonal properties of these functions and expands Mehler's conal function in terms of normal Lamé functions of complex degree. *J. A. Todd.*

***Watson, G. N. A Treatise on the Theory of Bessel Functions.** Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944. vi+804 pp. $15.00.

The second edition of this important treatise is hardly more than a reprint of the first edition. Some misprints were eradicated but no attempt was made to deal systematically with the progress in the field since 1922. One important exception is the reference to Siegel's discovery that $J_\nu(z)$ is transcendental when ν is rational and z is algebraic, $z\neq 0$. *G. Szegö* (Stanford University, Calif.).

Olevsky, M. N. On a generalization of Bessel functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 5–10 (1943). [MF 11182]

In a space of constant curvature k the equation

$$(1)\quad x^2(1-kx^2)y''+x(1-2kx^2)y'+(\mu^2x^2-p^2)y=0$$

plays the same role as Bessel's equation in a Euclidean space [M. Olevsky, C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 282–287 (1941); these Rev. **5**, 98]. In the present paper the author states many properties of the solutions of (1). Proofs are to follow in a later paper. *F. G. Dressel.*

Rutgers, J. G. Extension d'une série des fonctions de Bessel, due à Lommel, et de quelques séries des fonctions de Bessel analogues. I, II. Nederl. Akad. Wetensch., Proc. **45**, 929–936, 987–993 (1942). [MF 10444, 10450]

As a generalization of the Lommel series

$$\sum_{n=0}^{\infty}(-1)^n(\nu+2n)J_{\nu+2n}(x)=\tfrac{1}{2}xJ_{\nu-1}(x),$$

the author considers the sum

$$(1)\qquad S_{\nu,m}(x)=\sum_{n=0}^{\infty}(-1)^n(\nu+2n)^mJ_{\nu+2n}(x)$$

for positive integral values of m. In part I it is shown that when m is odd (1) can always be expressed as a finite series and detailed formulas are given for several values of m and ν. In part II even values of m are considered, but in this case (1) is represented in closed form only for integral values of ν. A short discussion of the related series

$$S'_{\nu,m}(x)=\sum_{n=0}^{\infty}(\nu+2n)^mJ_{\nu+2n}(x)$$

is also included. *M. C. Gray* (New York, N. Y.).

Chaundy, T. W. An extension of hypergeometric functions. I. Quart. J. Math., Oxford Ser. **14**, 55–78 (1943). [MF 9932]

The author investigates functions of the form

$$y=\sum_{n=0}^{\infty}\gamma_nF_nx^n,$$

where γ_n is a gamma-quotient

$$\frac{\Gamma(A_1+n)\Gamma(A_2+n)\cdots}{\Gamma(C_1+n)\Gamma(C_2+n)\cdots},$$

and F_n a terminating hypergeometric function, usually with unit argument,

$${}_rF_s\left[\begin{matrix}-n, a_2+h_2n, \cdots, a_r+h_rn\\ c_1+k_1n, \cdots, c_s+k_sn\end{matrix}\right],$$

the h and the k being integers. Such functions satisfy a differential equation

$$\sum_{m=0}^{p}x^mf_m(\delta)y=0,\qquad \delta\equiv xd/dx,$$

and this equation is said to be of rank p. By a transformation, consisting in expanding F_n and changing the order of summation in the repeated sum, many extended hypergeometric functions can be expressed in terms of the familiar hypergeometric functions.

Several types of extended hypergeometric functions are described and examples of the simplest types, together with their differential equations, are treated in detail. The author also discusses the "Burchnall phenomenon," that is, the fact that the order and rank of the differential equation satisfied by an extended hypergeometric function are not uniquely determined: the order of the equation can be decreased at the cost of increasing the rank and vice versa. For this reason, the idea of rank cannot be transferred from the differential equation to the functions satisfying it. In this connection the eliminants of two differential equations are discussed. By eliminating dyalitically in two different ways, the eliminant of least rank and the eliminant of least order (the latter is the eliminant usually so called) can be obtained. *A. Erdélyi* (Edinburgh).

Mohan, B. On confluent hyper-geometric functions. Proc. Benares Math. Soc. (N.S.) **4**, 59–60 (1943). [MF 10351]

The author deals with the particular case of confluent hypergeometric functions in which, in Whittaker's well-known notation [Modern Analysis, chap. 16], $k=m$. In the first part of the paper a list is given of formulae derived from well-known ones by putting $k=m$. A critical reader may take exception to $(-1)^{m-\frac{1}{2}}$ appearing in relations meant to hold for all values of m. Also, the last formula of this section seems to be incorrect. In the second part of the paper the author writes down the well-known confluent hypergeometric equation for the case discussed by him and integrates this differential equation by means of definite integrals. The integrals obtained are essentially the well-known ones. *A. Erdélyi* (Edinburgh).

Shanker, Hari. On some integrals and expansions involving Whittaker's confluent hyper-geometric functions. Proc. Benares Math. Soc. (N.S.) **4**, 51–57 (1943). [MF 10350]

A typical series expansion is

$$W_{k+n,m}(z)=(-1)^n\Gamma(m+k+n+\tfrac{1}{2})n!\times\sum_{r=0}^{n}\frac{(-1)^rz^{r/2}W_{k+r/2,m+r/2}(z)}{(n-r)!r!\Gamma(m+k+r+\frac{1}{2})},\quad \Re(\tfrac{1}{2}-k+m)>0,$$

while the integrals obtained are of the type

$$\int_0^\infty\frac{e^{-t^2/4}t^{2k-1}D_{2n}(t)dt}{(t^2+z^2)^{k-\mu+\frac{1}{2}}}=\frac{\pi^{\frac{1}{2}}\Gamma(2k)e^{z^2/4}z^{2\mu-1}}{2^{k-\mu}\Gamma(k-\mu+\frac{1}{2})}D_{-2k}(z),\quad \Re(k)>0,$$

$\frac{1}{2}+k-\mu$ not a negative integer. *M. C. Gray.*

Pasricha, B. R. Some infinite integrals involving Whittaker functions. Proc. Benares Math. Soc. (N.S.) **4**, 61–69 (1943). [MF 10352]

The author evaluates the integrals

$$\int_0^\infty y^{2\rho-1}e^{-(\alpha^2+\frac{1}{2})\alpha y^2}W_{k,\nu}(sy)^2\phi(y)dy,$$

where $\phi(y)$ is either ${}_0F_3(l_1+1, l_2+1, l_3+1; -t^2y^2)$ or ${}_0F_4(l_1+1, \cdots, l_4+1; ty^4)$. These integrals are evaluated by expanding $\phi(y)$ in powers of y, integrating term by term, rewriting the result as an iterated sum and interchanging the order of summation. A great number of particular cases, especially of the first integral, are written out in detail. Some of the particular cases are integral representations for

$$\frac{(x/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\,{}_0F_2(m+1, n+1; -(x/3)^3).$$

The author calls this function a Humbert function. This name seems misleading if a new name is necessary; the name "Bessel functions of the third order," adopted by Humbert himself, is preferable, it being indicative of the way in which these functions arise in connection with differential equations. Generalizations of the above integrals are also given in which $\phi(y)$ is any hypergeometric function of the variable ty^3 or ty^4, respectively. *A. Erdélyi* (Edinburgh).

Bateman, H. Some integral relations. Bull. Amer. Math. Soc. **50**, 745–749 (1944). [MF 11289]

First, the author shows that the unique solution of

$$u(r)=\int_0^{\pi/2}\cos(kr\sin a)\,U(r\cos a)da$$

is given by

$$U(s)=(2/\pi)\left[u(0)'\operatorname{ch}(ks)+s\int_0^{\pi/2}\operatorname{ch}(ks\sin b)u'(s\cos b)db\right].$$

Next he discusses the asymptotic behavior [cf. W. R. Burwell, Proc. London Math. Soc. (2) **22**, 57–72 (1924)] of the recently tabulated function [cf. G. Grimminger, J. Franklin Inst. **236**, 413–443, 509–520 (1943); these Rev. **5**, 194]

$$F(x)=\int_0^\infty e^{-m^{4/4}}\cos mx\,dm,$$

which can be inferred by the theory of Haar, in virtue of the relation

$$\int_0^\infty e^{-s^4}F(t^{\frac{1}{2}})t^{-\frac{1}{2}}dt=(\pi s^{-\frac{1}{2}}/2)\exp(-s^{-\frac{1}{2}}/4),\quad \Re(s)>0,$$

from the behavior of the last function at $s=0$. He shows how the last relation can be deduced in a new way and applies all this to functions V_0, V_1, V_2 and V_3 defined by him previously [cf. Bateman, Partial Differential Equations of Mathematical Physics, Cambridge University Press, 1932]. *František Wolf* (Berkeley, Calif.).

Baber, T. D. H. and Mirsky, Leonid. Note on certain integrals involving Hermite's polynomials. Philos. Mag. (7) **35**, 532–537 (1944). [MF 11343]

A table is constructed of the integral

$$I_t(m, n, s)=\int_{-\infty}^\infty \xi^t\psi_m(d^t\psi_n/d\xi^t)d\xi$$

for $t=0, 1, 2$; $s=0(1)6$ and $m-n=0(1)6$, n being a non-negative integer. The functions ψ_m are orthogonal functions for the range of integration and are defined by the equations

$$\psi_m(\xi)=(2^m\cdot m!(\pi/\alpha)^{\frac{1}{2}})^{-\frac{1}{2}}e^{-\frac{1}{2}\alpha\xi^2}H_m(\xi\alpha^{\frac{1}{2}}),$$
$$e^{2\xi\tau-\tau^2}=\sum_{m\geq 0}(\tau^m/m!)H_m(\xi).$$

An auxiliary integral

$$J_s(m, n)=\int_{-\infty}^\infty H_m(\xi)H_n(\xi)e^{-\xi^2}\xi^s d\xi$$

is first evaluated and series are obtained from which a series for $I_0(n+k, n, s)$ is obtained, values of $I_t(m, n, s)$ for $t>0$ being calculated with the aid of a reduction formula.

An alternative method of calculating the series for $J_s(m, n)$ was suggested by P. J. Daniell and is given in an appendix. A generating function for $J_s(n+k, n)$ is given in a footnote but no mention is made of the fact that this indicates at once that there is consequently a number of expressions in terms of the hypergeometric function. A symmetrical expression worthy of study is derived from

$$J_s(m, n)=\frac{\Gamma(\frac{1}{2})m!n!s!}{p!q!r!}F(p-s, q-s; r+1; \tfrac{1}{2})2^r,$$

$2p=m-n+s$, $2q=n-m+s$, $2r=m+n-s$, $m-n=k$, by using terms of type $\Gamma(r+h)$ in the denominators and retaining only those terms for which $r+k$ is a positive integer. The integral $J_s(n+k, n)$ may be calculated also from a result given by C. Strachan [Proc. Roy. Soc. London. Ser. A. **158**, 591–605 (1937)]. *H. Bateman* (Pasadena, Calif.).

Hadwiger, H. Über eine Formel mit einem speziellen Differentialoperator. Comment. Math. Helv. **15**, 353–357 (1943).

For the operator $(1/x)d/dx$ the following formula is shown to hold:

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n F(x)=(-2x^2)^{-n}\sum_{\nu=1}^n\frac{(2n-\nu-1)!}{(n-\nu)!(\nu-1)!}(-2x)^\nu F^{(\nu)}(x).$$

The above formula is then used to develop several relations between generalized Laguerre polynomials and Bessel functions. Also statements of the following type are proved:

$$\sum_{n=0}^\infty(-1)^n x^n J_{n+\frac{1}{2}}(x)\frac{\xi^n}{n!}=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\frac{\sin x(1+2\xi)^{\frac{1}{2}}}{(1+2\xi)^{\frac{1}{2}}}.$$

F. G. Dressel (Durham, N. C.).

*Lindman, C. F. **Examen des Nouvelles Tables d'Intégrales Définies de M. Bierens de Haan. Amsterdam 1867.** G. E. Stechert & Co., New York, 1944. 231 pp. $7.50.

This is a reprint of a paper (containing additions and corrections) that appeared in the Kongl. Svenska Vetenskaps-Akademiens Handlingar, Vol. **24**, No. 5. Stockholm, 1891.

Differential Equations

*Ince, E. L. **Ordinary Differential Equations.** Dover Publications, New York, 1944. viii+558 pp. $3.75.

Reprint of the English edition which appeared in 1926 at Longmans, Green and Co., London.

Terracini, Alejandro. On certain systems of differential equations of third order. Anais Acad. Brasil. Ci. **16**, 227–244 (1944). (Spanish) [MF 11500]

Continuation of the author's papers in Univ. Nac. Tucumán. Revista A. **2**, 245–329 (1941) and **3**, 195–234 (1942); these Rev. **4**, 54, 256.

Sawyer, W. W. A property of certain differential equations. Quart. J. Math., Oxford Ser. **15**, 34–39 (1944). [MF 10687]

Les dérivées successives des solutions d'une équation différentielle linéaire du second ordre à coefficients polynomes sont les solutions d'autres équations analogues dans lesquelles le degré des coefficients va généralement en croissant. Il s'ensuit que l'équation des dérivées n-ièmes présente en général des singularités apparentes. On établit les deux théorèmes suivants. (1) Si le degré du premier coefficient est le même pour tous les n, l'équation possède au plus 3 singularités effectives, dont une peut être à l'infini. (2) Les singularités apparentes sont les racines d'une équation dont les coefficients sont des polynomes en n du même degré que leur poid. *B. Levi* (Rosario).

Niemytzki, V. Intégration qualitative du système $dx/dt = Q(x, y)$, $dy/dt = P(x, y)$ en première approximation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 190–192 (1943). [MF 9242]

In a previous paper [same C. R. (N.S.) **38**, 62–65 (1943); these Rev. **4**, 276] the author developed a countable procedure for approximating the integral curves of a system of the type indicated in the title. In the present paper he points out that in a certain sense a critical stage of the procedure can be defined. If the construction is interrupted before the critical stage has been reached the essential topological features of the system are obliterated. On the other hand, the continuation of the procedure beyond the critical stage brings only a quantitative refinement. The precise formulation of this result is somewhat complicated. Proofs are omitted in the paper. *W. Hurewicz.*

Bellman, Richard. A stability property of solutions of linear differential equations. Duke Math. J. **11**, 513–516 (1944). [MF 110°7]

The solutions of

$$F(f) \equiv u''(x) + g(x)u'(x) + f(x)u(x) = 0$$

are shown to be stable in the following sense. If, for one bounded function $f_1(x)$, all solutions of $F(f_1) = 0$ belong to $L^p(0, \infty)$ and $L^q(0, \infty)$ $(1 < p \leqq 2 \leqq q,\ 1/p + 1/q = 1)$, then all solutions of $F(f_2) = 0$, where $f_2(x)$ is any bounded function, belong to $L^p(0, \infty)$ and $L^q(0, \infty)$. *F. G. Dressel.*

Bulgakov, B. V. On operational solutions of systems of linear differential equations with constant coefficients. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 234–238 (1943). [MF 11068]

Let $f_{jk}(D)$ and $g_{jm}(D)$ be polynomials in the differential operator d/dt. The author considers the equations

$$\sum f_{jk}(D)x_k(t) = \sum g_{jm}(D)y_m(t), \quad j = 1, \cdots, n,$$

where summation is on k and m. The $y_m(t)$ are known functions. Using matrix notation and the Laplace transform the author develops explicit formulae for $x_k(t)$ subject to any initial conditions at $t = 0$. He is interested in applications to automatic control problems. *N. Levinson.*

Theodorchik, K. Contribution to the theory of synchronization of relaxatory autooscillations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 54–57 (1943). [MF 11174]

An interesting analysis is given which reveals how the phenomenon of synchronization operates in the case of a relaxatory oscillator which is being modulated by a signal of period nearly equal to $1/N$ times the free period of the oscillator, where N is an integer. The method has the decided advantage of not depending on the number of degrees of freedom of the system. The method makes clear why, when the strength of the modulating signal is gradually increased, the syncronization bands at first broaden, and then, as the signal strength reaches a certain critical value, a breakdown occurs. *N. Levinson.*

Parodi, Hypolyte. Détermination des zones de vitesses critiques des systèmes de transmission par bielles. C. R. Acad. Sci. Paris **217**, 338–339 (1943). [MF 11102]

In the case of a moving train, for example, certain oscillatory systems are described by differential equations with periodic coefficients, where the period T of the coefficients is inversely proportional to the velocity V of the train. The question then arises for what ranges of values of T (or, what is equivalent, for what ranges of V) are the solutions of the differential equation stable and for what ranges are they unstable. For the second order equation

$$y'' + P(2\pi t/T)y' + Q(2\pi t/T)y = 0,$$

with $P(\tau)$ and $Q(\tau)$ of period 2π, an explicit discussion is given. The s, such that there is a solution $y(t)$ with $y(t+T) = sy(t)$, is a root of a quadratic equation. For those ranges of T for which a root of the quadratic equation exceeds one in magnitude the system is unstable. [Unless the average of $P(\tau)$ is zero, the author's method of finding where $s = 1$ for determining these ranges is wrong; and even where the average of $P(\tau)$ is zero it is necessary to find the points where $s = -1$ also. No references to the extensive mathematical literature on this subject are given.]

N. Levinson (Cambridge, Mass.).

Pfeiffer, G. V. Equations, linear in Jacobians, on which are distributed generalized-Jacobi systems of linear equations in many functions, and the symbolic forms of the canonical type connected with them. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 364–365 (1943). [MF 11075]

The generalized-Jacobi system

$$(1) \qquad p_{\sigma j} = Z_\sigma{}^j - \sum_{i=1}^{q} X_i{}^j p_{\sigma i},$$

$p_{ij} = \partial z_i/\partial x_j$; $j = q+1, \cdots, n$; $\sigma = 1, \cdots, k$, is said to be distributed on the equation linear in Jacobians

$$(2) \qquad J_n \equiv \sum_{s=0}^{m} \sum_{\sigma_1 \cdots \sigma_s}^{k} \sum_{\tau_1 \cdots \tau_s}^{n} M_{\tau_1 \cdots \tau_s}^{\sigma_1 \cdots \sigma_s} \frac{\partial(Z_{\sigma_1} \cdots Z_{\sigma_s})}{\partial(x_{\tau_1} \cdots x_{\tau_s})} = 0,$$

if the substitution of the p_{ij} defined by (1) into equation (2) makes J_n identically zero. The author finds the form of J_n if (1) is distributed on it. *F. G. Dressel.*

van der Kulk, W. Zur Theorie der verallgemeinerten Pfaff'schen Gleichungen. Nederl. Akad. Wetensch., Proc. **45**, 26–31 (1942). [MF 10374]

A system of generalized Pfaff equations in a space of n dimensions is defined by

$$\overset{i}{F}(\xi^k, d_1\xi^{[\rho_1} \cdots d_m\xi^{\rho_m]}) = 0, \quad i = d+1, \cdots, m(n-m),$$

where the $\overset{i}{F}$ are homogeneous in the $\binom{m}{i}$ expressions $d_1\xi^{[\rho_1}\cdots d_m\xi^{\rho_m]}$. The author announced a theorem concerning the integrability of this system, which is analogous to the theorem of Cartan-Kähler for the case that the equations of Pfaff are linear and homogeneous in $d_1\xi^{[\rho_1}\cdots d_m\xi^{\rho_m]}$. Proof for $m=2$ has been given in previous papers [Nederl. Akad. Wetensch., Proc. **44**, 452–463, 625–635 (1941); these Rev. **3**, 43]. For $m=n-1$ the theorem reduces to the theorem of Jacobi for homogeneous partial differential equations with one unknown variable. *D. J. Struik.*

Schouten, J. A. und van der Kulk, W. Beiträge zur Theorie der Systeme Pfaffscher Gleichungen. Nederl. Akad. Wetensch., Proc. **45**, 624–629 (1942). [MF 10425]

In previous papers [Schouten and van der Kulk, same Proc. **43**, 179–188, 453–462, 674–686 (1940); these Rev. **2**, 54; van der Kulk, same Proc. **44**, 452–463, 625–635 (1941); these Rev. **3**, 43] the authors proved their principal theorem, which states that a field of simple covariant q-vectors of semi-rank ρ has g linearly independent vector divisors of class $2\rho+1$. This paper modifies a proof of this theorem presented by E. Cartan [these Rev. **3**, 123] and in this way establishes the principal theorem for a general field $\mathfrak{S}_q^{n-1}$. Such a field is a system of ∞^d $(n-1)$-directions defined in the local E_n of a general manifold X_n, defined by $B_b(\xi^\kappa, w_\lambda)=0$; $\kappa, \lambda=1, 2, \cdots, n$; $b=1, \cdots, m$; $m=n-1-d$, with rank of $\partial B_b/\partial w_\kappa$ equal to m. This theorem is used to determine all factors of all possible classes of a given field of simple covariant q-vectors, and to obtain a system of arithmetical invariants for the classification of fields of q-vectors. A given system of Pfaff equations can now be reduced to its simplest equivalent form. *D. J. Struik.*

Mitra, Samarendra Kumar. A new method of solution of the boundary value problems of Laplace's equation relating to two spheres. I. Bull. Calcutta Math. Soc. **36**, 31–39 (1944). [MF 11214]

This paper contains a discussion of a new method of determining the potential due to two charged spheres. It is an elaboration of a discussion outlined by the author in an earlier paper [Science and Culture **9**, 397–398 (1944); these Rev. **5**, 204]. The potential is given by an infinite series whose terms depend upon two infinite sets A_m and B_m, $m=0, 1, 2, \cdots$, of undetermined constants. From a study of the functions $f(z)=\sum_{m=0}^{\infty}A_m z^m$ and $g(z)=\sum_{m=0}^{\infty}(-1)^m B_m z^m$ the values of the A's and B's are obtained. On substituting these in the series and making certain simplifications, the author obtains the desired result. *F. W. Perkins.*

Wasow, Wolfgang. Asymptotic solution of boundary value problems for the differential equation $\Delta U+\lambda\partial U/\partial x=\lambda f(x, y)$. Duke Math. J. **11**, 405–415 (1944). [MF 11076]

Let $u(x, y; \lambda)$ be the solution of the differential equation

$$\Delta u+\lambda\partial u/\partial x=\lambda f(x, y),$$

taking on prescribed values on the boundary of a domain B. Under certain continuity conditions on $f(x, y)$ and the boundary of B, the author shows that the limiting function

$$\lim_{\lambda\to\infty} u(x, y; \lambda)$$

exists and satisfies the equation $\partial u/\partial x=f(x, y)$. The limiting function will in general take on the prescribed boundary values only on part of the boundary of B.

F. G. Dressel (Durham, N. C.).

Weinstein, A. and Rock, D. H. On the bending of a clamped plate. Quart. Appl. Math. **2**, 262–266 (1944). [MF 11134]

In a series of publications since 1937 Weinstein has developed a method for the solution of boundary value problems concerning plates by representing these solutions as limits of solutions of problems involving only the differential equations for membranes. A remarkable feature of Weinstein's method is that the solutions of the approximating membrane problems not only converge themselves to the solution of the plate problem but that the convergence also holds for the derivatives in the interior. The present paper shows in detail how the method applies to the problem of bending of a clamped (uniformly loaded square) plate.

R. Courant (New York, N. Y.).

Lowan, Arnold N. Note on Green's functions in the theory of heat conduction. Philos. Mag. (7) **35**, 495–498 (1944). [MF 11300]

The author derives Green's functions (G) for problems in heat conduction for the region $0<x<\infty$, $0<y<\infty$ with boundary conditions (a) $G=0$ for $x=0$, $G=0$ for $y=0$; (b) $\partial G/\partial x=0$ for $x=0$, and $\partial G/\partial y=0$ for $y=0$; (c) $G=0$ for $x=0$, $\partial G/\partial y=0$ for $y=0$. Some Green's functions for heat conduction problems in three dimensions are also considered. *A. E. Heins* (Cambridge, Mass.).

Jaeger, J. C. Some problems involving line sources in conduction of heat. Philos. Mag. (7) **35**, 169–179 (1944). [MF 10866]

With the aid of the Laplace transform, the author calculates the temperature distribution for the following problems. (A) The temperature due to a unit instantaneous line source at zero time, parallel to the z axis, through the point whose cylindrical coordinates are (r', θ'), in a composite solid, for which the region $0\leqq r<a$ has conductivity K_1, density ρ_1, specific heat c_1 and diffusivity $\mathfrak{D}_1=K_1/c_1\rho_1$, and the region $r>a$ has the corresponding quantities K_2, ρ_2, c_2, $\mathfrak{D}_2$. (B) A line source, emitting Q heat units per unit length per unit time, moves round the surface of a cylinder of radius a and thermal constants K, $\mathfrak{D}$ with a uniform angular velocity ω. There is no loss of heat from the surface. Generalizations of these problems are discussed.

A. E. Heins (Cambridge, Mass.).

Vernotte, Pierre. Valeur, à l'extérieur d'un système matériel, de la série de Fourier représentant une distribution initiale de températures à l'intérieur de ce système. C. R. Acad. Sci. Paris **217**, 364–366 (1943). [MF 11106]

Let $F(x)$ be defined in $0\leqq x\leqq l$ by the trigonometric series

$$F(x)=\sum_i A_{ni}\cos(n_i x/l-\varphi_{ni})=P(x)+Q(x),$$

where $n_1<n_2<n_3<\cdots$ are the roots of a transcendental equation arising in a certain problem of linear heat conduction and

$$P(x)=\sum_i A_{ni}\cos\varphi_{n_i}\cos(n_i x/l),$$

$$Q(x)=\sum_i A_{ni}\sin\varphi_{n_i}\sin(n_i x/l).$$

As pointed out by the author in a former note [C. R. Acad. Sci. Paris **206**, 590–592 (1938)] the calculation of the extension of $F(x)$ for all x in $(-\infty<x<\infty)$ is of importance for the solution of the corresponding heat conduction

problem with initial values $F(x)$ for small values of the time. In the present paper this extension problem is taken up with a different method under the assumption that $n_i \operatorname{cotg} \varphi_{n_i}$ is a rational function of n_i^2. This method consists essentially in finding in an interval free of the singularities of F a linear differential equation with constant coefficients of order three for P and Q. *E. H. Rothe.*

Shvez, M. On the problem of unstationary distribution of air temperature near the underlying surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 144–147 (1943). [MF 11168]

The influx of radiant energy to unit earth surface consists of long wave radiation a directed towards the underlying surface and of short-wave radiation s. If z is the vertical coordinate, t the time and $f(z, t)$ the influx of heat due to thermic radiation, then the temperature T in the boundary layer satisfies the equation

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+f(z, t).$$

In the underlying medium the temperature T_1 satisfies

$$\frac{\partial T_1}{\partial t}=\frac{\partial}{\partial z}\left(k_1\frac{\partial T_1}{\partial z}\right).$$

The boundary conditions at the underlying surface ($z=0$) are taken as

$$-\lambda\frac{\partial T}{\partial z}+\lambda_1\frac{\partial T_1}{\partial z}+\sigma T^4=a+s-v, \quad T(0, t)=T_1(0, t).$$

Here k, k_1, λ, λ_1 are coefficients of conductivity, σ the constant of Stephan-Boltzmann and v the flux of heat lost on evaporation. In the present paper $f(z, t)$ is assumed zero, $k=cz^n$, k_1 constant. Under these conditions an approximate solution to the above problem is obtained through the use of integral equations of the first kind. *F. G. Dressel.*

Minakshisundaram, S. Studies in Fourier Ansatz and parabolic equations. J. Madras Univ. **14**, 73–142 (1942). [MF 10979]

This paper studies parabolic equations in two independent variables. Chapters I and II lay the foundation for the developments. The author uses the concept of correlation to treat axiomatically the theory of a special type of abstract vector field called a linear correlation space. After developing a Hausdorff topology the author introduces the idea of the norm $\|f\|$ of an element f, and shows that if a linear correlation space is metrisable it is a Banach space. Chapter III deals with the equation $u_{xx}-u_t=0$ with the boundary conditions (1) $u(0, t)=u(2\pi, t)$, $t>0$, (2) $u_t(0, t)=u_x(2\pi, t)$, $t>0$, and (3) $\lim_{t\to 0} u(x, t)=u_0(x)$, where $u_0(x)$ is integrable L in $(0, 2\pi)$ and periodic with period 2π, and so has a formal Fourier series $u_0(x)\sim\sum A_n(x)$. The author discusses various properties of the well-known formal solution

$$u=\sum_{n=0}^{\infty} A_n(x)e^{-n^2t}.$$

Chapter IV contains a discussion of the equation

$$u_{xx}-u_t=f(x, t). \tag{4}$$

Let R be the domain $0\leqq x\leqq 2\pi$, $0\leqq t\leqq T$, and let R_1 be any rectangular domain in the interior of R with sides parallel to those of R centered at the point (x, y). Denote by S_1 the boundary and by $|R_1|$ the area of R_1. If $f(x, t)$ is integrable in R and if $u(x, t)$ is a solution of (4) in R, then

$$\iint_{R_1} f(x, t)dxdt=\int_{S_1} udx+\int_{S_1}(\partial u/\partial x)dt, \tag{5}$$

whence

$$\partial^2 u/\partial x^2-\partial u/\partial t = \lim_{|R_1|\to 0}\int_{S_1} u(\xi, \tau)d\xi+\int_{S_1}(\partial u/\partial\xi)d\tau=f(x, t), \tag{6}$$

almost everywhere in R. Given such an $f(x, t)$, any $u(x, t)$ which satisfies (5) in every $R_1<R$ is said to be a solution of (4) in R in the sense of (5). It is also desirable in some cases to replace boundary condition (2) by

$$\int_{t_1}^{t_2} u_x(0, t)dt=\int_{t_1}^{t_2} u_x(2\pi, t)dt. \tag{7}$$

If (7) is satisfied by every pair (t_1, t_2) such that $0<t_1<t_2\leqq T$, then (2) is satisfied almost everywhere on $0<t<T$. In (3), $u_0(x)$ is now required to be differentiable; any restriction on the space of $f(x, t)$ is assumed to apply also to $u_0(x)$. The following theorems are given. (i) If $f(x, t)$ is continuous in every R and periodic with respect to x there exists a solution of (4) in the sense of (5) satisfying the boundary conditions (1), (2) and (3). The solution is given by an absolutely convergent Fourier series in x. Moreover, u_x is continuous and periodic. (ii) If $f(x, t)$ is bounded and measurable in every R there exists a solution of (4) in the sense of (5) satisfying the conditions (1), (3) and (7). The solution is given by an absolutely convergent Fourier series. Also u_x is a bounded measurable function continuous in x for every t for which (2) is valid. Results are also obtained concerning properties of certain functions related to the solution, under various restrictions imposed on $f(x, t)$.

Chapter V of the paper is reviewed below.

F. W. Perkins (Hanover, N. H.).

Minakshisundaram, S. Studies in Fourier Ansatz and parabolic equations. J. Indian Math. Soc. (N.S.) **6**, 41–50 (1942). [MF 10980]

This is chapter V of the memoir reviewed above. It deals with the nonlinear equation

$$u_{xx}-u_t=f(x, t, u, u_x), \tag{1}$$

subject to the boundary conditions (2) $u(0, t)=u(2\pi, t)$, $u_x(0, t)=u_x(2\pi, t)$ and $u(x, 0)=u_0(x)$. In connection with the first two boundary conditions it is obviously intended that the restriction $t>0$ stated explicitly in the earlier paper should be retained. Using the invariant point theorem of Schauder [Studia Math. **2**, 171–180 (1930)] the author gives results which may be summarized as follows. (i) If the function $f(x, t, u, u_x)$ is independent of u_x, is uniformly continuous with respect to u and satisfies the inequality

$$|f(x, t, u, u_x)|\leqq\varphi(t+|u|),$$

where $\varphi(\gamma)$ is a positive monotonic increasing function vanishing at the origin, then (1) has a solution satisfying (2) for sufficiently small t. If the hypothesis is valid when $\varphi(t+|u|)$ is replaced by $\varphi(t)$ we may infer the existence of a solution for all finite t. (ii) If $f(x, t, r, s)$ is continuous with respect to all the variables, is periodic with respect to x, satisfies a Hölder condition with respect to x, r and s, and if further

$$\|f(x, t, u, u_x)\|\leqq\varphi(t+|u|+|u_x|),$$

then there exists for small t a solution of (1) satisfying (2).

Some modifications of this theorem are also discussed. (iii) If the function $f(x, t, r, s)$ is continuous and satisfies the inequality

$$|f(x, t, r, s)| \leqq \varphi(t+|r|+|s|),$$

then there exists for small t a solution of (1) satisfying (2). By suitably modifying the inequality the existence of a solution for all t is assured. (iv) If we supplement the hypotheses of each of the preceding theorems by the requirement that the bounded measurable function $f(x, t, r, s)$ be a monotonic increasing function of r independently of the other variables, then the solution is in each case unique.

F. W. Perkins (Hanover, N. H.).

Bergman, Stefan. Solutions of linear partial differential equations of the fourth order. Duke Math. J. **11**, 617–649 (1944). [MF 11095]

Let

$$P(z, z^*, w; E, f) = \int_{-1}^{1} E(z, z^*, t) f(\tfrac{1}{2}w(1-t^2))dt/(1-t^2)^{\frac{1}{2}},$$

$z=x+iy$, $z^*=x-iy$. The author has previously shown [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 1169–1198 (1937)] that the solutions of the partial differential equation with analytic coefficients a, b, c:

$$(1) \qquad \Delta u+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0$$

can be represented in the form

$$u(x, y) = \Re[P(z, z^*, z; E, f)].$$

Here $f(z)$ is an arbitrary analytic function of z, and E is a particular solution of a differential equation depending on (1). The present paper carries this type of analysis to fourth degree partial differential equations. It is shown that, if M, L, N, A, D, C are analytic functions of z and z^*, the solutions of

$$(2) \quad U_{zzz^*z^*}+MU_{zz}+LU_{zz^*}+NU_{z^*z^*}+AU_z+DU_{z^*}+CU=0$$

can be represented in the neighborhood of the origin in the form

$$U(z, z^*) = \sum_{i=1}^{2} P(z, z^*, z; E_i, f_i)+\sum_{i=1}^{2} P(z, z^*, z^*; E_{2+i}, g).$$

Here $f(z)$, $g(z)$ are analytic functions of z in the neighborhood of $z=0$; E_1, E_2 satisfy a differential equation depending on (2), and E_3, E_4 satisfy another differential equation depending on (2). The last part of the paper is devoted to a discussion of functions of the types $P(z, z^*, z; E, f)$ and $P(z, z^*, z^*; E, f)$.

F. G. Dressel (Durham, N. C.).

Functional Analysis

Markouchevitch, A. Sur les bases (au sens large) dans les espaces linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 227–229 (1943). [MF 11066]

The author wishes to define a generalized base which will include as a particular case the trigonometric functions in spaces L and C. If E is a space of type F, in Banach's terminology, the author calls a set of elements $\{e_\alpha\}$ of a space E a strongly linearly independent set if, whenever a sequence $\sum_{j=1}^{s} A_j^{(n)} e_{\alpha_j}$ of linear combinations of the e_α converges to the zero of E, then, for each fixed j, $A_j^{(n)} \to 0$ as $n \to \infty$. A base (in the sense of Schauder) [Banach, Théorie des opérations linéaires, Warsaw, 1932, p. 110] is necessarily a strongly linearly independent set, but not conversely. The author states three necessary and sufficient conditions for a set to be strongly linearly independent, one being that there is a system $\{L_\alpha\}$ of linear functionals such that $L_\alpha(e_\beta)=\delta_{\alpha\beta}$, that is, a biorthogonal set of linear functionals. The set $\{e_\alpha\}$ is said to form a base (in the broad sense) if it is fundamental, strongly linearly independent and has the property that $L_\alpha(x)=0$ for every α implies that x is the zero of E [L_α is defined above]. A base in the broad sense attaches to every element of E a generalized Fourier series which has many properties analogous to those of the Fourier series of continuous or integrable functions. Although it is not known whether a separable Banach space necessarily has a countable base (in the sense of Schauder), it always has a countable base (in the broad sense); the author indicates how a proof of this fact can be given by imbedding E in C. If $\{e_n\}$ is a base (in the broad sense) for a space E of type F, and $\{f_n\}$ is a sequence of elements of E such that, for every e in E,

$$\|\sum L_n(e)(e_n-f_n)\| \leqq \lambda\|e\|, \qquad 0<\lambda<1,$$

then $\{f_n\}$ is also a base (in the broad sense).

R. P. Boas, Jr. (Cambridge, Mass.).

Day, Mahlon M. Uniform convexity in factor and conjugate spaces. Ann. of Math. (2) **45**, 375–385 (1944). [MF 10275]

The author considers some relationships between the notion of uniform convexity and a sort of dual notion of uniform flattening. A Banach space is said to be uniformly flattened if there exists a function η positive for $0<\epsilon\leqq 2$ such that $\lim_{\epsilon\to 0}\eta(\epsilon)=0$ while

$$(2-\|b_1+b_2\|)/\|b_1-b_2\| \leqq \eta(\epsilon)$$

if $\|b_1\|=\|b_2\|=1$ and $\|b_1-b_2\|\leqq\epsilon$. The function η is called a modulus of flattening. The principal theorem is that a Banach space B is uniformly convex (flattened) if and only if B^* is uniformly flattened (convex) and a modulus in one space is determined by that in the other. As a corollary it follows that a complete uniformly flattened space is reflexive. For two dimensional spaces B a formula giving a modulus of convexity (flattening) for B in terms of a modulus of flattening (convexity) in B^* is developed. For general spaces B it is shown that B is uniformly convex (flattened) if and only if there is a common modulus of convexity (flattening) for all the factor spaces of B and a common modulus of convexity (flattening) for all the two dimensional factor spaces of B.

N. Dunford.

Lorch, Edgar R. The structure of normed Abelian rings. Bull. Amer. Math. Soc. **50**, 447–463 (1944). [MF 10739]

The paper, an address, is concerned with the highly important notion of a space as a ring instead of as a group. The spaces considered are normed Abelian rings, that is, R is a Banach space as well as an Abelian ring with unit e such that $|a\cdot b|\leqq|a|\cdot|b|$. The author surveys the significant contributions by himself and others to the study of these spaces. The fundamental formula in the study of the reducibility of a ring is the Cauchy formula

$$j=(1/2\pi i)\int_C dz/(ze-a),$$

where a is fixed and C is a simple rectifiable curve in the complex plane, which avoids entirely the spectrum of a. This spectrum is defined as the set of complex numbers λ for which $(a-\lambda e)^{-1}$ does not exist. It is indicated that a

ring R is irreducible if and only if the spectrum of each of its points is a connected set. Gelfand's analysis of the maximal ideals of a normed ring is discussed; the quasi-nilpotents in R are defined as those elements a for which line $(\mu a)^n=0$ for every complex μ. These elements are characterized by the fact that their spectrum is the one point $\lambda=0$, and they constitute the radical of R, that is, the intersection of all maximal ideals. The role of analytic functions in R is described and the use of ideals in the study of these functions is emphasized. The last several sections are devoted to rings of analytic functions and of infinitely differentiable functions. *H. H. Goldstine.*

Rickart, C. E. An abstract Radon-Nikodym theorem. Trans. Amer. Math. Soc. **56**, 50–66 (1944). [MF 10790]

Let $\mathfrak{F}$ be a countably additive Boolean algebra with greatest element M, let m be a (real (possibly infinite) completely additive) measure function on $\mathfrak{F}$ and let $\mathfrak{X}$ be a Banach space with weak topology v. A multi-valued function x on $\mathfrak{F}$ to $\mathfrak{X}$ is said to be contractive whenever $e\subseteqq e'$ implies $x(e)\subseteqq x(e')$. In terms of v a definition of v-integrability (and the v-integral $X(e)=\int_e x(s)dm(s)$) of a contractive function x is given essentially following Garrett Birkhoff [Ann. of Math. (2) **38**, 39–56 (1937)]. After obtaining several preliminary results the following generalization of the Radon-Nikodym theorem is proved. If M is the join of a countable number of elements of finite measure and X is any completely additive function with values in $\mathfrak{X}$ and defined for elements in $\mathfrak{F}$ having finite measure, then a necessary and sufficient condition for $e\in\mathfrak{F}$ that $X(e)$ be representable as a v-integral is that X be absolutely continuous.

In the last section of the paper $\mathfrak{F}$ is taken to be a σ-field of subsets of an abstract set M, where M is the union of countably many disjoint sets of finite measure. Let $L(M)$ be the Banach space of real valued, summable point functions. The author indicates membership in $L(M)$ by $f(\cdot)\in L(M)$ and designates by $f(:)$ the contractive function on $\mathfrak{F}$ defined for each $e\in\mathfrak{F}$ by saying that $f(e)$ is the set of values $f(p)$ as p ranges over e. Note that, for $f(\cdot)\in L(M)$,

$$\|f(\cdot)\|=\int_M |f(s)|dm(s)$$

and, for $x(:)$ a v-integrable function on $\mathfrak{F}$ to $\mathfrak{X}$, define

$$\|x(:)\|=\operatorname{l.u.b.}_{0<m(e)<\infty}(1/m(e))\left\|\int_e x(s)dm(s)\right\|.$$

The following representation theorem is proved. Each bounded linear transformation T on $L(M)$ to $\mathfrak{X}$ has a v-integral representation of the form

$$Tf(\cdot)=\int_M x(s)f(s)dm(s), \qquad f(\cdot)\in L(M),$$

where $x(:)$ is v-integrable on every set of finite measure and $\|T\|=\|x(:)\|$. *J. F. Randolph* (Oberlin, Ohio).

Mackey, George W. Equivalence of a problem in measure theory to a problem in the theory of vector lattices. Bull. Amer. Math. Soc. **50**, 719–722 (1944). [MF 11283]

Let S denote an arbitrary set, $\mathfrak{F}$ the set of real valued functions $f(s)$ defined on S. Let $P(f)$ be a linear functional on $\mathfrak{F}$. For s in S, $P_s(f)=f(s)$ is called a point functional; P is called bounded if its values on every bounded subset of $\mathfrak{F}$ are bounded. The main theorem states that a real valued bounded P is a finite linear combination of point functionals unless there is a completely additive two-valued measure α on S which vanishes for points. It follows from a result of Ulam that, if the cardinal of S is not too large, then P must be a linear combination of point functionals. The proof is based on the consideration of nonnegative functionals, that is, those P's for which $f\geqq 0$ implies $P\geqq 0$. On page 721, 14 lines from below, the words "does not vanish" should be replaced by "vanishes." *F. J. Murray* (New York, N. Y.).

Millsaps, Knox. Differential calculus in topological groups. I. Revista Ci., Lima **44**, 485–492 (1942). [MF 8750]

Millsaps, Knox. Differential calculus in topological groups. II. Revista Ci., Lima **45**, 45–52 (1943). [MF 8754]

The author defines two types of differential for a function f with domain in a rather general topological group G, the range being a second topological group. The differentials are generalizations, to less restricted groups, of Michal's differential [Revista Ci., Lima **43**, 155–176 (1941); these Rev. **3**, 50. Reference is also made to a paper by Michal to appear in the same Revista]. Each differential $f(x_0, \delta x)$ is a homomorphism for δx belonging to the central subgroup of G and is a first order approximation to the increment of f at x_0, in a sense defined in terms of topological and algebraic operations on the groups. It is shown, for each of the two types of differential, that the differentials of all orders are unique, that the existence of the differential implies continuity of the function and that a differentiable function of a differentiable function is differentiable. *J. L. Kelley* (Aberdeen, Md.).

Cohen, L. W. On linear equations in Hilbert space. Bull. Amer. Math. Soc. **50**, 729–733 (1944). [MF 11285]

This paper is concerned with the solution in Hilbert space of the system of equations $y_i=\sum_{j=1}^{\infty}a_{ij}x_j$; $i=1, 2, \cdots$. The method is to solve m of these equations and let $m\to\infty$. The author gives sufficient conditions that the equations should have a solution for every set of y's with $\sum_{i=1}^{\infty}|y_i|^2<\infty$. These conditions are based on a process involving Schwartz's inequality and thus are subject to the limitations usually associated with the notion of "finite norm." For example, the author's conditions will not show that the system $y_i=x_i$ always has a solution. Furthermore, it seems to the reviewer that necessary and sufficient conditions are readily obtainable from what is known concerning the structure of transformations. [For a discussion of the relation between operator structure and equations, see the expository paper of the reviewer in J. Math. Phys. Mass. Inst. Tech. **22**, 148–157 (1943); these Rev. **5**, 100.] *F. J. Murray.*

Julia, Gaston. Remarques géométriques sur le problème des moments dans l'espace hilbertien. C. R. Acad. Sci. Paris **216**, 257–260 (1943). [MF 10958]

The author considers the moment problem in a Hilbert space H of solving the equations $(x_k, x)=\xi_k$, $k=1, 2, \cdots$, where (x_k) is a sequence of independent vectors in H and ξ_k are complex numbers. By simple geometric arguments he shows that the solution x^n in V_n, the linear closed manifold determined by $x_1, \cdots, x_n$, of the reduced problem $(x_k, x^n)=\xi_k$, $k=1, \cdots, n$, converges strongly to x, the solution of the moment problem, if and only if the sequence

$$M_n=\|x^n\|=\max\,|\textstyle\sum^n\lambda_k\xi_k|\cdot\|\sum^n\lambda_k x_k\|^{-1},$$

for all λ_k, is uniformly bounded; that is, he establishes the

Riesz characterization of a solution of the moment problem. In the case that the moment problem is solvable for all $\xi=(\xi_1, \xi_2, \cdots)$ in H he shows that the Schmidt condition is equivalent to that of Riesz and hence that the operator $\lim M_n(\xi)$ is bounded and linear. The paper closes with an application of the above results to the problem of determining the dual (y_k) of a system (x_k), that is, of finding a sequence y_k in the linear closed manifold determined by $x_1, x_2, \cdots$ such that $(x_i, y_j)=\delta_{ij}$. *H. H. Goldstine.*

Julia, Gaston. Détermination des adjoints de quelques opérateurs linéaires non bornés de l'espace hilbertien. C. R. Acad. Sci. Paris **216**, 221–224 (1943). [MF 10955]

The author considers a sequence of vectors A_i in a Hilbert space H, an orthonormal base e_i of H and the function $Ax \equiv \sum x_i A_i$, where $x=\sum x_i e_i$. The operator A is considered as being defined over the domain d_A (d_A') for which $\sum x_i A_i$ converges strongly (weakly). It is shown that the adjoint of A when defined over d_A or d_A' is given by $A^*y=\sum e_i(A_i, y)$ and is defined over the set of all y such that $\sum|(A_i, y)|^2$ converges. *H. H. Goldstine.*

Rothe, Erich. On non-negative functional transformations. Amer. J. Math. **66**, 245–254 (1944). [MF 10570]

Let E be a Hilbert space, let r be a positive number, let S be the sphere $\|x\|=r$, let a be a point of S and T the tangent plane to S at a, that is, T has the equation $x=a+\lambda b$, where $\|b\|=1$, $(b, a)=0$. Stereographic projection of S on T from $-a$ is defined, notions of convexity are introduced and these notions are used in conjunction with a theorem of Schauder [Studia Math. **2**, 171–180 (1930)] to establish the following theorem. Let s be a closed subset of S which is convex with respect to $-a$ and let $F_1(x)$ be defined for each $x \epsilon s$, where (i) F_1 is completely continuous, (ii) the image of s is contained in s. Then there exists at least one fixed-point of F_1. This theorem is used to establish an eigenvalue theorem which in turn is given two interpretations in Hilbert space L^2. The paper concludes with several remarks concerning the application of existence proofs for eigen-values of integral equations. In the introduction an unsolved problem in linear integral equations is stated. *J. F. Randolph* (Oberlin, Ohio).

Neumark, M. A. On a representation of additive operator set functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 359–361 (1943). [MF 11073]

Consider a function $B(\Delta)$ for which the variable Δ runs over a system $\mathfrak{S}$ of subsets of a set X and B is a positive Hermitian operator in a Hilbert space $\mathfrak{H}$ of unspecified dimensions; B is additive relative to Δ and $B(0)=0$, $B(X)=1$. The main result of this brief but valuable note is that there is a space $\mathfrak{H}'$ and a resolution of the identity $E(\Delta)$, for $\Delta \epsilon \mathfrak{S}$ in $\mathfrak{H}'$, such that $B(\Delta)x=PE(\Delta)x$ for P the projection of $\mathfrak{H}'$ on $\mathfrak{H}$ and $x \epsilon \mathfrak{H}$. If X is the interval $0 \leqq t \leqq 1$, $\mathfrak{S}$ the set of measurable sets on this interval, then $\mathfrak{H}'$ is a kind of cross product space of $\mathfrak{H}$ and L_2, in which the usual construction has been modified by using a norm which depends on the operators $B(\Delta)$. *F. J. Murray.*

Neumark, M. A. On spectral functions of a symmetric operator. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 285–296 (1943). (Russian. English summary) [MF 10942]

This paper is a continuation of a previous investigation by the author [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 53–104, 277–318 (1940); cf. these Rev. **2**, 104, 105 for terminology]. In the present paper the set of all spectral functions associated with a given closed symmetric operator is specified. This is done in terms of the integral

$$\int_{-\infty}^{\infty} \frac{dE_1(u)g}{u-\lambda}$$

and an application is made to the moment problem. *F. J. Murray* (New York, N. Y.).

Rellich, Franz. Störungstheorie der Spektralzerlegung. V. Math. Ann. **118**, 462–484 (1942). [MF 10710]

[The last paper of the series appeared in Math. Ann. **117**, 356–382 (1940); these Rev. **2**, 105.] Suppose a self-adjoint operator $A(\epsilon)$ depending analytically on a parameter ϵ has a discrete spectrum for $\epsilon=0$. As the author had proved earlier under suitable conditions, to every characteristic value λ_n and characteristic element φ_n of $A(0)$ there exist characteristic values and functions $\lambda_n(\epsilon)$, $\varphi_n(\epsilon)$ of $A(\epsilon)$ which depend analytically on ϵ. The question arises whether the set of $\lambda_n(\epsilon)$ and $\varphi_n(\epsilon)$ represent the total spectrum of $A(\epsilon)$. The main theorem of the paper states that this is the case if the operators $A(\epsilon)$ are self-adjoint in a common subspace $\mathfrak{A}$ of the Hilbert space $\mathfrak{H}$, and if $A(\epsilon)u$ is regular in ϵ for every u in $\mathfrak{A}$. Counterexamples are given to show that without restriction the statement would not be true. It is proved, however, that the statement would be true under a modified condition which essentially only requires that the forms $(u, A(\epsilon)u)$ be semi-bounded in a common subspace of $\mathfrak{H}$. Applications to integral operators are given.

The following theorems concerning the existence of a discrete spectrum are added. (1) If for the subspace $\mathfrak{A}$ an operator exists which is self-adjoint in the domain $\mathfrak{A}$ and has a discrete spectrum, then every operator self-adjoint in $\mathfrak{A}$ or a subspace of $\mathfrak{A}$ has a discrete spectrum. (2) Let the bounded symmetric operator R have the range $\mathfrak{W}$. If a completely continuous symmetric operator exists whose range contains $\mathfrak{W}$, then R is also completely continuous. *K. O. Friedrichs* (New York, N. Y.).

Hedlund, Gustav A. Sturmian minimal sets. Amer. J. Math. **66**, 605–620 (1944). [MF 11398]

Let E be the space consisting of all sequences of type $\omega^*+\omega$ of two symbols, with the Baire metric, and let T denote the transformation that displaces the indices of each sequence by one place. The author defines and studies the properties of a certain class of minimal sets (in the sense of G. D. Birkhoff) for this system. The orbits which make up these sets are symbolic trajectories of a type previously studied by the author and M. Morse [Amer. J. Math. **62**, 1–42 (1940); these Rev. **1**, 123] and called Sturmian. It is shown that the sets are minimal under all nonzero powers of T, and that each contains a pair of asymptotic orbits. The latter property is especially interesting because it is shown to have the following implications: (i) T is not almost periodic on the minimal set, although it is locally almost periodic; (ii) the powers of T are not equicontinuous; (iii) the orbits of the set are not equivalent under orbit preserving homeomorphisms of the set. Property (iii) throws light on a conjecture of Birkhoff concerning the homogeneity of recurrent motions. *J. C. Oxtoby* (Bryn Mawr, Pa.).

RELATIVITY

Karapetoff, Vladimir. Special theory of relativity in hyperbolic functions. Rev. Modern Phys. **16**, 33–52 (1944). [MF 10766]

The author considers many of the well-known results of special relativity and shows the advantages of using hyperbolic functions in this theory. *M. Wyman.*

Scott-Iversen, P. A. Introductory notes on a reformulation of the special theory of relativity. Philos. Mag. (7) **35**, 105–120 (1944). [MF 10861]

The author desired to reformulate the mechanics of the special theory of relativity so that the units of length, time and mass become invariant and simultaneity becomes absolute. Unfortunately the paper is incomplete due to the death of the author. For inertial frames of reference the reformulation is carried out and the special relativity effects are obtained by postulating an anisotropic distribution for the velocity of light. *M. Wyman* (Edmonton, Alta.).

Birkhoff, George D. Flat space-time and gravitation. Proc. Nat. Acad. Sci. U. S. A. **30**, 324–334 (1944). [MF 11335]

The author discusses the fundamental postulates of his theory. Matter is characterized by a stream vector (u^i) in the flat space-time of special relativity. This leads to the form $T^{ij}=\rho u^i u^j-p(\rho)g^{ij}$ for the energy-tensor. The requirement that the maximum velocity of matter shall not be greater than that of light leads to the equation of state $p=\frac{1}{3}\rho$. A force-vector f^i is defined by $f^i=\partial T^{i\alpha}/\partial x^\alpha$ and it is assumed that $f^i=\rho^3\varphi^i_{\alpha\beta}u^\alpha u^\beta$. By analogy with Newtonian gravitational theory it is also assumed that the $\varphi^i_{\alpha\beta}$ are linear homogeneous functions of the first derivatives of the gravitational potentials defined by an extension of the Poisson equation. Some consequences of the theory are discussed briefly. *G. C. McVittie* (London).

Barajas, A., Birkhoff, G. D., Graef, C. and Vallarta, M. Sandoval. On Birkhoff's new theory of gravitation. Phys. Rev. (2) **66**, 138–143 (1944). [MF 11154]

Replies are made to criticisms by Weyl [these Rev. **4**, 285] of G. D. Birkhoff's theory of relativity [Proc. Nat. Acad. Sci. U. S. A. **29**, 231–239 (1943)]. The differential equations of this theory are briefly developed and a cosmological term is added to the gravitational potentials. The formula for the advance of the periastron in the two-body problem is stated and it is suggested that the formula may have an application to the motion of double-stars. *G. C. McVittie* (London).

Weyl, Hermann. Comparison of a degenerate form of Einstein's with Birkhoff's theory of gravitation. Proc. Nat. Acad. Sci. U. S. A. **30**, 205–210 (1944). [MF 10907]

The author criticizes G. D. Birkhoff's theory of relativity in flat space by pointing out that it does not make inertial and gravitational mass equal. He does this by comparing it with the form which Einstein's theory takes when the first approximation to the theory is regarded as being true to all degrees of approximation. It is also shown that, both in G. D. Birkhoff's theory and in this "degenerate" form of Einstein's, the deflection of light by the sun, the displacement of the perihelion of Mercury and the gravitational red-shift of spectral lines can only be obtained after rather forced ad hoc assumptions. *G. C. McVittie* (London).

Wilson, W. Kinematic relativity. Philos. Mag. (7) **35**, 241–249 (1944). [MF 10856]

This paper contains a criticism of E. A. Milne's kinematical relativity. In particular, attempts are made to prove that Milne's equations of motion of a free particle in the presence of a substratum are not Lorentz invariant. These equations are:

$$\frac{1}{y}\frac{d\vec{V}}{dt}=-\frac{1}{x}(\vec{P}-\vec{V}t), \tag{1}$$

where $\vec{P}$ stands for the space coordinates, $\vec{V}$ for the velocity of the particle, $X=t^2-\vec{P}^2/c^2$, $Y=1-\vec{V}^2/c^2$. The author's proof is based upon the assumption that $\vec{V}t$ is the space component of a vector. This assumption is wrong, which can be seen clearly by writing

$$V_s t=\frac{dx}{ds}\cdot\frac{ds}{dt}t, \quad ds^2=c^2dt^2-dx^2-dy^2-dz^2.$$

Now dx/ds is a component of a vector, but $ds/dt\cdot t$ is a ratio of two time components of two different vectors and of course is not invariant with respect to a Lorentz transformation. Only if $d\vec{V}/dt=0$, is $\vec{V}t$ the space component of a vector $(t, \vec{V}t)$. *L. Infeld* and *A. Schild.*

Milne, E. A. On the nature of universal gravitation. Monthly Not. Roy. Astr. Soc. **104**, 120–136 (1944). [MF 10867]

Presidential address to the Royal Astronomical Society.

Schroedinger, E. The affine connexion in physical field theories. Nature **153**, 572–575 (1944). [MF 10757]

Schrödinger, Erwin. The earth's and the sun's permanent magnetic fields in the unitary field theory. Proc. Roy. Irish Acad. Sect. A. **49**, 135–148 (1943). [MF 10030]

The author's unitary field theory (U.F.T.) equations [Proc. Roy. Irish Acad. Sect. A. **49**, 43–58 (1943); these Rev. **5**, 165] are solved for the case of a centered dipole neglecting gravitation and other charges. It is shown that the solution thus obtained differs from that of the Gaussian dipole field. It is remarked that the earth's magnetic field also differs from the latter. The theoretical difference is identified with the experimental one and the constant μ entering into the theory is evaluated. The U.F.T. associates with each vector potential a current proportional to the former; the magnitude of this current is computed and compared with experimental evidence. This evidence is very weak and A. Schmidt and J. Bartels [Gerland's Beiträge zur Geophysik **55**, 292 (1939)] are quoted as saying the currents are spurious. Nevertheless the author points to the agreement between the sign of the measured currents and those of the computed ones, which are shown to be small. The relation between the theory and what is known of the permanent magnetic field of the sun is also discussed. *A. H. Taub* (Princeton, N. J.).

Schrödinger, Erwin. The point charge in the unitary field theory. Proc. Roy. Irish Acad. Sect. A. **49**, 225–235 (1944). [MF 10751]

The author continues his studies of a general unitary field theory as initiated by him in a previous paper [same

Proc. Sect. A. **49**, 43–58 (1943); these Rev. **5**, 165]. In this paper the solution of the field equations for a field due to a point charge is obtained on the basis of the new theory and the physical implications of this solution are discussed. When the point charge is embedded in spherical space the corresponding solution possesses only one singularity. This implies that the universe as a whole need no longer be electrically neutral. *M. Wyman* (Edmonton, Alta.).

Schrödinger, Erwin. Unitary field theory: conservation identities and relation to Weyl and Eddington. Proc. Roy. Irish Acad. Sect. A. **49**, 237–244 (1944). [MF 10750]

The author compares the relationship between the affine connection Γ^k_{lm} and the metrical connection $\{^k_{lm}\}$ as given by his own general unitary field theory [same Proc. Sect. A. **49**, 43–58 (1943); these Rev. **5**, 165] with the same relationship as given by the older theories due to H. Weyl and A. S. Eddington. It is suggested that the new theory does not possess Weyl's gauge invariance because one of the fundamental fields (the meson field) has not as yet been considered. The author intends to introduce this missing field and this point will be discussed further in a subsequent paper. The more general relationship between the above connections as given by Eddington is shown to contain both the Schrödinger and Weyl relationships as special cases. *M. Wyman* (Edmonton, Alta.).

Schrödinger, Erwin. The union of the three fundamental fields (gravitation, meson, electromagnetism). Proc. Roy. Irish Acad. Sect. A. **49**, 275–287 (1944). [MF 10753]

The author proceeds with his attempt to generalize his former unified field theory [same Proc. Sect. A. **49**, 43–58 (1943); these Rev. **5**, 165] so as to include the gravitational, electromagnetic and meson fields. A linear connection Δ^k_{lm} is considered for which the usual symmetry condition is replaced by a slightly weaker condition. The Einstein-tensor is obtained in the usual way from the curvature tensor R^i_{klm} and in addition a new six-vector is obtained by contraction with respect to (i, k). Assuming the Lagrangian function to be a function of both the Einstein-tensor and this six-vector the field equations are obtained by means of a variation principle.

A discussion is given about the physical roles to be assigned to quantities entering into the field equations and it is pointed out that at the present stage of the theory these roles are not strictly defined. The author indicates that a further generalization, obtained by dropping the assumed condition for the linear connection, may clear up this point and may also permit the restoration of gauge invariance. *M. Wyman* (Edmonton, Alta.).

McConnell, James and Schrödinger, Erwin. The shielding effect of planetary magnetic fields. Proc. Roy. Irish Acad. Sect. A. **49**, 259–273 (1944). [MF 10752]

The motion of a charged particle in a magnetic field due to a celestial body is considered on the basis that the cosmical term μ of Maxwell's equations is not zero. In this way the authors obtain a generalization of Störmer's results for auroral theory. Taking the value of $1/\mu$ to be 30,000 km (a value obtained by considering the magnetic field at the surface of the earth), various numerical results are obtained for the earth and sun. *M. Wyman* (Edmonton, Alta.).

Eddington, Arthur S. The representation of space curvature in quantum theory. Revista Ci., Lima **46**, 13–28 (1944) = Actas Acad. Ci. Lima **6**, 220–235 (1943). [MF 10759]

All physical coordinates and observables must be measured relative to some physical origin which is itself liable to an uncertainty of position relative to a conceptual mathematical frame of reference. It is postulated that the physical origin used in current quantum theory is the centroid of a large number of particles. The Bernoulli fluctuation for the difference between the actual and the mean number of particles in a given volume is analyzed into two fluctuations, one arising from the finiteness of the mean number of particles and the other from the finiteness of the total number of particles. From this the "radial" and "transverse" standard deviations of space measurements along and transverse to the "radius vector" are calculated. When these are combined into the line-element of differential geometry, the metric of spherical space is obtained. The connection between the standard deviations occurring in the expression of this metric and the constants of atomic physics is obtained through the hypothesis that an ultimate standard of length must be constructable at any place and time from a recorded specification. This specification can only be a set of pure numbers and therefore can only be obtained by using a quantum-specified structure. Hence the constants in the metric of space-time must be derivable from the atomic constants and, when this is done, the velocity of recession of the spiral nebulae can be calculated, amongst other quantities. The upper limit to the speed is found to be 572 km/sec/mgparsec which compares with the observed value of 560 km/sec/mgparsec.

G. C. McVittie (London).

Komischke, A. On the relations between light and gravitation. Publ. Inst. Mat. Univ. Nac. Litoral **4**, 167–201 (1943). (Spanish. English summary) [MF 10698]

The hypothesis is made that a photon moving in a gravitational field has a velocity v and a mass m which are related to its velocity c and mass m_0 in free space by the equation $m=m_0(c/v)^\alpha$, where α is an undetermined constant. The motion of the photon under Newton's inverse square law of gravitation is worked out and it is shown that, by taking $\alpha=7/5$, the deflection of light by the sun can be increased from 1.75″, predicted by Einstein's theory, to 2.19″. The red-shift of the spectral lines in the light of the spiral nebulae can also be worked out starting from the same hypothesis. Finally, the wave equation with a phase-velocity $c(1-R/r)^{\frac{1}{2}}$, where R is the sun's radius, is solved and it is shown that an original mass of gas around the sun will tend to collect at certain distances from the sun. A selection from amongst these distances is in rough agreement with Bode's law of planetary distances.

G. C. McVittie (London).

Lichnerowicz, André. Sur l'invariant intégral des équations relativistes de l'électromagnétisme. C. R. Acad. Sci. Paris **216**, 863–865 (1943). [MF 10657]

The author gives two integral invariants of the electromagnetic equations of relativity. *M. Wyman.*

Wataghin, G. Relativity and supplementary indeterminacy. Anais Acad. Brasil. Ci. **16**, 57–58 (1944). [MF 10892]

MECHANICS

Chen, Pei-Ping. Dyadic analysis of space rigid framework. J. Franklin Inst. **238**, 325–334 (1944). [MF 11365]

The author outlines a general method of analyzing three-dimensional frames. His two basic vector relations are nothing but the vector components of the motor relation which R. von Mises used in his general analysis of space structures [Z. Angew. Math. Mech. **4**, 193–213 (1924), eq. (24)]. A direct method of structural analysis based on these vector relations is developed. Except for particularly simple structures, this method will, however, require the solution of an undesirably large number of simultaneous linear equations. An iterative process is therefore developed which is patterned on the model of the moment distribution method of H. Cross. *W. Prager* (Providence, R. I.).

Goldberg, Michael. New five-bar and six-bar linkages in three dimensions. Trans. Amer. Soc. Mech. Eng. **65**, 649–661 (1943). [MF 11198]

A linkage is an assemblage of rigid bodies or links, certain pairs of which are hinged, which means that all points of a certain line fixed with respect to the one link are also fixed with respect to the other. The hinges at the ends of a link are two skew lines which have a certain mutual angle and (shortest) distance, called the twist and length of the link. The Bennett linkage [Proc. London Math. Soc. (2) **13**, 151–173 (1914), in particular, pp. 152–153] has four links and four hinges, corresponding to the sides and vertices of a "skew parallelogram," that is, a skew quadrilateral $ABCD$ whose opposite sides are equal, say $AD=BC=a$ and $AB=CD=b$; consequently its opposite angles are equal, say $A=C=\theta$ and $B=D=\varphi$. This figure is symmetrical by a half-turn about the join of the midpoints of the remaining edges (AC and BD) of the tetrahedron $ABCD$. Hence the tetrahedron has equal dihedral angles, say α, at the two edges of length a, and equal dihedral angles, say β, at those of length b. It follows that

$$\tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\varphi = \cos \tfrac{1}{2}(\alpha+\beta)/\cos \tfrac{1}{2}(\alpha-\beta).$$

The lines of the hinges at A, B, C, D are, of course, perpendicular to the respective planes DAB, ABC, BCD, CDA; hence the equal links AD, BC have equal twists α, and the equal links AB, CD have equal twists β.

The author constructs several new linkages by combining two or three Bennett linkages in various ways. His paper is followed by a lengthy discussion, in the course of which A. E. R. de Jonge finds these constructions exceedingly important. However, de Jonge has some adverse criticism as well; for example, he points out that the author gives an incorrect version of Bennett's formula displayed above. Actually, the critic's version of the same formula is again incorrect and both errors could have been detected immediately by considering the special case when the tetrahedron $ABCD$ is regular (so that $\theta=\varphi=\frac{1}{2}\pi$ and $\alpha=\beta=\arccos \frac{1}{3}$). *H. S. M. Coxeter* (Toronto, Ont.).

Campbell, J. W. Motion with respect to moving axes. Amer. Math. Monthly **51**, 377–381 (1944). [MF 11143]

The author uses the term Newtonian function to designate a function of the vector r from a point O to a point P, of certain of the time derivatives of the vector r' from a fixed point O' (with which O is instantaneously coincident) to P, and of the time. He derives expressions for the time derivative of a Newtonian function in terms of the derivative when O is fixed and of the velocity of O with respect to O'. Some examples and generalizations are discussed. *L. A. MacColl* (New York, N. Y.).

***Whittaker, E. T. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.** Fourth Ed. Dover Publications, New York, 1944. xiv+456 pp. $3.50.

Reprint of the fourth (latest) English edition which appeared in 1937 at the Cambridge University Press.

Manley, R. G. Roots of frequency equations. Nature and distribution of roots. Aircraft Engrg. **16**, 203 (1944). [MF 10902]

An attempt to prove in a simple way what the author considers to be a known important theorem: that a positive definite symmetric matrix has only distinct characteristic roots. The proposition is false. *W. Feller.*

Jouguet, Émile. Remarques sur les vitesses critiques et la stabilité séculaire des systèmes gyroscopiques. C. R. Acad. Sci. Paris **217**, 254–257 (1943). [MF 11109]

This is virtually identical with a previous note by the same author [C. R. Acad. Sci. Paris **214**, 929–931 (1942); these Rev. **5**, 132]. *L. A. MacColl* (New York, N. Y.).

Mohanty, Ramanath and Patnaik, Braja Behari. Vibration of air-columns in pipes. Bull. Calcutta Math. Soc. **36**, 79–82 (1944). [MF 11354]

This note deals with the free vibrations of a system consisting of a column of air in a frictionless pipe, which is open at both ends, and which is fitted with n thin movable pistons, each of mass M and each controlled by a spring of strength μ. The equation determining the natural frequencies of the system is derived, and the modes of motion corresponding to the several frequencies are investigated. *L. A. MacColl* (New York, N. Y.).

Astronomy

Ideljson, N. I. Periodic solutions of the problem of three bodies in Newton's "Principia." Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 177–182 (1944). (Russian) [MF 11125]

Subbotin, M. F. Sur une méthode pour augmenter la convergence des séries trigonométriques fondamentales pour la mécanique céleste. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 302–305 (1943). [MF 11178]

The author deals in particular with the development of

$$\Delta^{-1} \equiv (1-2\alpha \cos H+\alpha^2)^{-\frac{1}{2}} = \sum_{-\infty}^{+\infty} a_k \cos kH,$$

with $a_{-k}=a_k$. Writing $z=e^{iH}$, the problem is to choose a new variable w, such that

$$\sum_{-\infty}^{+\infty} a_k z^k = \sum_{-\infty}^{+\infty} A_k w^k,$$

with $A_{-k}=A_k$, with the requirement that the convergence be increased. The transformation proposed is, in its most

general form,

$$\prod_{n=1}^{m}\left(\frac{w-\alpha_n}{1-\alpha_n w}\right)=\prod_{n=1}^{m}\left(\frac{z-\beta_n}{1-\beta_n z}\right).$$

Special cases considered are $m=1$ and $m=2$. For $m=2$, with $\beta_1=\beta_2$, the transformation is identical with that used by Legendre [Traité des fonctions elliptiques, vol. II, 1826, p. 555]. With $\beta_1\neq\beta_2$, so chosen that the moduli of the singular points inside the circle $|w|=1$ are equal, the transformation is shown to be practically as effective as the transcendental substitution employed by Gyldén [Bull. Acad. Sci. St. Petersburg **14**, 195–231 (1870)].

D. Brouwer (New Haven, Conn.).

Rosenblatt, A. On the movement of a cosmic cloud. Amer. J. Math. **66**, 268–280 (1944). [MF 10572]

The present paper generalizes Sundman's researches on the n-body problem [Acta Soc. Sci. Fennicae **35**, no. 9, 27 pp. (1909); Acta Math. **36**, 105–179 (1913)] to a continuous distribution of matter (the "cosmic cloud") in equilibrium under its own gravitation. In particular, it is shown that a cosmic cloud cannot tend to its center of mass with its moment of inertia tending to zero simultaneously unless its total angular momentum is identically zero. Inequalities analogous to Sundman's inequality [see papers by G. García reviewed in these Rev. **4**, 57, 174] are also established. *S. Chandrasekhar* (Williams Bay, Wis.).

García, Godofredo. On a generalization of Newton's and Kepler's law to explain classically the secular motion of the perihelia of the planets, the determination of the period of the anomalistic revolution and the sidereal revolution. Revista Ci., Lima **46**, 219–280 (1944) =Actas Acad. Ci. Lima **7**, 163–224 (1944). (Spanish) [MF 10881]

The author proposes a correction to the Newtonian law of attraction by addition of a term proportional to r'/r^3, where r is the distance between two planets and r' its time derivative. The resulting differential equation for motion of one body with respect to another is then integrated by the method of successive approximations. The result of the second approximation is then used as an approximate solution. For this solution there is found to be an advance in perihelion proportional (in first approximation) to the square of the mean velocity, as in the relativistic theory. For proper choice of the proportionality constant in the correction term of the force law, this advance in perihelion then agrees with that predicted by relativity.

W. Kaplan (Providence, R. I.).

García, Godofredo. On the regularization of two dimensional three-body problem. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 3–12 (1943). (Spanish) [MF 11197]

Cf. the author's four extensive studies in Revista Ci., Lima **45** (1943); these Rev. **5**, 17.

García, Godofredo. On the actual state of the solar system. New dissipative forces acting in addition to universal gravitation. The problem of three bodies within a dissipative system. Actas Acad. Ci. Lima **7**, 10 pp. (1944). (Spanish) [MF 11196]

García, Godofredo. Celestial mechanics and its evolution. Revista Ci., Lima **45**, 145–164 (1944). (Spanish) [MF 11208]

García, Godofredo. On the law of density in nebulae as a function of the distance. Revista Ci., Lima **46**, 3–11 (1944) =Actas Acad. Ci. Lima **7**, 3–11 (1944). (Spanish) [MF 10495]

The author uses an approximate method to integrate Emden's differential equation [see R. Emden, Gaskugeln, Teubner, Leipzig, 1907, pp. 131–148; see also Enzyklopädie Math. Wiss. VI 2, 1926, pp. 404–412] for the density of a gravitational ideal gas at constant temperature. The result is in good agreement with the numerical integration of the same equation by Emden. *W. Kaplan.*

Roure, Henri. Sur une théorie analytique nouvelle des grosses planètes du système solaire. C. R. Acad. Sci. Paris **216**, 229–230 (1943). [MF 10957]

Graef Fernández, Carlos. Periodic orbits of the primary cosmic radiation. Bol. Soc. Mat. Mexicana **1**, no. 3, 1–31 (1944). (Spanish) [MF 10801]

Following the program initiated by Störmer, Vallarta and others [see, for example, M.S. Vallarta, An outline of the allowed cone of cosmic radiation, University of Toronto Studies, Appl. Math. Series, no. 3, Toronto, 1938] for the analysis of the orbits of cosmic ray particles moving in the earth's magnetic field, the author studies the motion of a particle in a rotating plane through the axis of the dipole. The trajectories are interpreted as extremals of a variational problem and hence as geodesics on a surface having positive curvature. Differential geometric properties of the trajectories are analyzed, and it is shown that, if a certain constant (analogous to an integration constant) is positive, then all periodic orbits cross the equator.

W. Kaplan (Providence, R. I.).

Roy, Sunil Kumar. A case of polytropic equilibrium configuration with variable angular velocity. Bull. Calcutta Math. Soc. **35**, 85–98 (1943). [MF 10736]

This paper discusses the distortions of a polytropic configuration of index $n=1$ rotating with an angular velocity depending on the distance from the axis of rotation. The methods used are standard ones and explicit solutions are obtained in terms of Bessel functions of half odd integral orders. *S. Chandrasekhar* (Williams Bay, Wis.).

Sen, N. R. On a problem in heat conduction in a sphere. Bull. Calcutta Math. Soc. **35**, 131–134 (1943). [MF 11010]

Stellar configurations with isothermal cores have many interesting properties. For example, it is known that for a given mass there is a limit to the amount of matter that can be contained in the isothermal core while the radius of the configuration itself passes through a succession of maxima and minima through the sequence [cf. L. R. Henrich and S. Chandrasekhar, Astrophys. J. **94**, 525–536 (1941) and M. Shoenberg and S. Chandrasekhar, Astrophys. J. **96**, 161–172 (1942)]. In the present paper certain aspects of this phenomenon are illustrated under much simplified conditions. Thus, considering a solid sphere in a thermally steady state, with a coefficient of conductivity varying as an inverse power of the distance from the center and containing an isothermal core on the surface of which a given constant amount Q of heat per unit area and per unit time is generated, the author shows that the radius R, at which the temperature attains some assigned value (say, zero) expressed as a function of the fraction x of the radius

occupied by the isothermal core, has a minimum. Again, under the same circumstances, if it be supposed further that heat is being lost by radiation at the outer surface R, then the ratio α^2 of the heat radiated per unit area and unit time from a spherical surface at the temperature of the isothermal core to the quantity Q enters the problem as a parameter. In particular, if $\alpha^2>1$, $0<x<1$ and $R(x)$ has a minimum while, if $\alpha^2>1$, $0<x<\alpha$ and $R(x)$ is monotonic decreasing. *S. Chandrasekhar* (Williams Bay, Wis.).

Wares, Gordon W. Partially degenerate stellar models. Astrophys. J. **100**, 158–175 (1944). [MF 11149]

In this paper stellar models are constructed in which the underlying equation of state is that of a gas obeying the laws of the Fermi-Dirac statistics. The equation of state is used in its exact form as given in certain tabulations of E. C. Stoner and J. McDougall [Philos. Trans. Roy. Soc. London. Ser. A. **237**, 67 (1939)]. Two classes of models are studied: (1) the "standard model" in which the ratio of the gas to the total pressure is assumed constant and (2) the isothermal model in which the temperature is assumed constant. The relevant equations of equilibrium are numerically integrated and the results exhibited in the form of tables and diagrams. *S. Chandrasekhar.*

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. Astrophys. J. **99**, 180–190 (1944). [MF 10161]

A solution of the equation of radiative transfer is sought which shall be of the form

$$I(\tau, \cos\theta)=\sum_{l=0}^{\infty} I_l(\tau)P_l(\cos\theta),$$

where τ is the optical depth and $P_l(\cos\theta)$ is a spherical harmonic. First, second and third approximations to this solution are obtained by retaining the first three, five and seven terms, respectively, in the above series for I. These approximations are worked out explicitly and the corresponding law of darkening derived. The validity of the method is checked by evaluating numerically Hopf's exact solution of the problem, which involves a definite integral. This shows that, except for θ very close to $\pi/2$, the second and third approximations give entirely satisfactory agreement with the exact solution and that the third approximation is substantially more accurate than the second. [Cf. the following review.] *G. C. McVittie* (London).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. II. Astrophys. J. **100**, 76–86 (1944). [MF 10776]

[Cf. the preceding review.] A method of solution of the equation of radiative transfer $\mu\, dI/d\tau=I-J$, where $J=\frac{1}{2}\int_{-1}^{+1}I d\mu$, is developed by using Gauss's formula for numerical quadratures. It provides a formal solution which can be written down at once for any order of approximation and is thus superior to the method used in the paper reviewed above. The first, second, third and fourth approximations are worked out in detail and the corresponding laws of darkening found. A by-product of the method of solution is a rigorous proof of the validity of the Hopf-Bornstein relation $J(0)=(\sqrt{3}/4)F$, where F is the net flux. Two further problems which can be solved on the same lines are the radiative equilibrium of a planetary nebula and the standard case in the theory of the formation of absorption lines. *G. C. McVittie* (London).

Kopal, Zdeněk. Some tables to facilitate computation of elements of eclipsing binaries. Proc. Amer. Philos. Soc. **88**, 145–149 (1944). [MF 10749]

More than thirty years ago Russell and Shapley developed the theory of eclipsing variable stars for components whose projected discs are circular and whose surface brightnesses are uniform or are radially symmetrical. Kopal has pointed out that modern observations with photoelectric photometers have produced light curves of several variables which are accurate to about 0.001 mag. Some of these light curves justify a refinement in the theory with appropriate allowance for tidal and rotational distortion in the shapes of the two stars and for the consequent nonuniform (gravitational) distribution of their surface brightnesses. In order to facilitate the computation of the elements of an eclipsing variable, Kopal has introduced a series of functions $I^m_{\beta,\gamma}$ defined by

$$\pi r_2{}^q I^m_{\beta,\gamma}=\int_{\delta-r_1}^{c}[r_2{}^2-(\delta-x)^2]^{\beta/2}[2\delta(s-x)]^{\gamma/2}[\delta-x]^m dx,$$

where r_1 and r_2 are the radii of the two stars, δ is the apparent distance between their centers in units of the radius of the relative orbit, $s=(r_1{}^2-r_2{}^2+\delta^2)/2\delta$ and $c=s$ or $\delta+r_2$ depending upon whether the eclipse is partial or annular. The superscript and the two subscripts of $I^m_{\beta,\gamma}$ are integers. When $\gamma=0$ the stars are uniformly bright; when $\gamma=1$ they are completely darkened at the limbs. Finally, $q=1+\beta+\gamma+m$. The use of these functions in actual computations was described by Kopal in earlier papers [Proc. Amer. Philos. Soc. **85**, 399–431 (1942); these Rev. **4**, 73 and other papers reviewed there]. The present paper gives three short tables, for uniformly bright stars and for darkened stars, respectively. The tables give the values of $I^m_{\beta,0}$ and of $J^m_{\beta,1}$ as functions of the argument α, where $\alpha=2\sin^{-1}k$, and $k^2=\frac{1}{2}(1-\mu)$ or $k^2=2/(1-\mu)$ and $\mu=(\delta-s)/r_2$. The tables are arranged in intervals of 20° for α, and are given to four decimal figures. The author recommends Everett's interpolation formula and lists for this purpose the second tabular differences M''. *O. Struve* (Williams Bay, Wis.).

McVittie, G. C. Models of the universe and cosmological time-scales. Nature **154**, 477 (1944). [MF 11345]

Hydrodynamics, Aerodynamics, Acoustics

v. Mises, R. Integral theorems in three-dimensional potential flow. Bull. Amer. Math. Soc. **50**, 599–611 (1944). [MF 11032]

If $\mathbf{q}(\mathbf{r})$ is the velocity in a potential flow, so that div $\mathbf{q}=0$ and curl $\mathbf{q}=0$, for what vector functions $\mathbf{f}(\mathbf{q}, \mathbf{r})$ is the equation

$$\int \mathbf{f}(\mathbf{q}, \mathbf{r})\cdot d\mathbf{S}=0 \tag{1}$$

correct if the integral is extended over the complete boundary of a region in which $\mathbf{f}$ has continuous derivatives of the first order with respect to the six components x, y, z of $\mathbf{r}$ and u, v, w of $\mathbf{q}$? The author deduces a general, ten-parametric expression for $\mathbf{f}$ to satisfy (1) and proceeds to obtain simple examples by choosing special values of the parameters. Next, the general result is applied to obtain a formula, analogous to Cauchy's in two dimensions, by which the value of $\mathbf{q}$ at a point within the region of regu-

larity is calculated from its values on the boundaries. This formula, in turn, yields a result which appears to be an extension of the Biot-Savart formula and gives the velocity induced by discontinuity surfaces. Further application of the general result leads to two formulas, analogous to Blasius' in two dimensions, which give the force and moment of the fluid reaction upon the rigid bodies enclosed in the region:

$$\mathbf{F}=(\rho/2)\int q^2\mathbf{n}'dS, \quad \mathbf{M}=(\rho/2)\int q^2(\mathbf{r}\times\mathbf{n}')dS.$$

Here $\mathbf{n}'$ is a unit vector symmetrical to the unit normal $\mathbf{n}$ with respect to the plane normal to $\mathbf{q}$, ρ is the fluid density, and the integration is carried over the surface enclosing the entire region. For the case of finite bodies immersed in a fluid moving with uniform velocity $\mathbf{c}$ at infinity, these formulas yield only a couple; this is d'Alembert's paradox. However, if one of the bodies or a discontinuity surface extends to infinity, a force is found:

$$\mathbf{F}=-\rho\mathbf{c}\times\mathbf{C}+(\rho/2)\int q'^2\mathbf{n}'dS,$$

where $\mathbf{C}$ is the "surface circulation" $\int(\mathbf{q}\times\mathbf{n})dS$, and $\mathbf{q}'$ denotes $\mathbf{q}-\mathbf{c}$. The first term is analogous to the Kutta-Joukowski formula; the second is analogous to one appearing in two-dimensional cases of flow with separation (Helmholtz flow). *W. R. Sears* (Inglewood, Calif.).

Poncin, Henri. Sur la construction des réseaux hydrodynamiques. C. R. Acad. Sci. Paris **214**, 816–818 (1942). [MF 9454]

The author applies his results concerning the use of functional analysis in hydrodynamics [same C. R. **213**, 341–342 (1941); these Rev. **5**, 192] to the determination of a flow in a channel, one of whose boundaries is a given curve, while the other may be either a given curve or a free boundary. This method is a generalization of the graphical method of successively constructed stream lines, where one starts from a boundary line and continues so as to render the potential and stream lines orthogonal to each other. From the fact that along one of the boundary curves the known hydrodynamical relations are fulfilled the author obtains a differential relation for the complex potential. This relation holds at first in the small; however, previously developed results on analytic continuation enable one to extend it to the large, and to determine the Riemann surface of the complex potential. The study of this surface yields the existence of "zones of instabilities" such that an analytic continuation beyond these zones is physically not significant. The author discusses the actual application of his method. The presentation is short and general so that many essential points are not explained. *S. Bergman.*

Serebrisky, J. M. Stream-line motion of fluids past bodies of revolution. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 150–153 (1943). [MF 11061]

In the calculation of the axial flow of an ideal incompressible fluid past bodies of revolution, the general solution for the velocity potential is usually obtained as a series involving Legendre polynomials and Legendre associated functions. A certain relation involving these functions and the unknown coefficients of the series must be satisfied on the surface of the body. The present author expands the associated Legendre functions and retains only the leading terms. This affords a great simplication and is said to be very accurate for slender bodies; for thicker bodies a method of correction by carrying more terms of the expansions is given.

In the case of transverse flow about a body of revolution a different relation is obtained at the surface [C. Kaplan, Nat. Adv. Comm. Aeronaut. Rep. no. 516 (1935)]. Again the author presents an approximation based on the series expansion of Legendre functions, which is simple and is said to be sufficiently accurate for practical purposes. The unknown coefficients in this case are related to those in the case of axial flow. *W. R. Sears* (Inglewood, Calif.).

Green, George. Solutions of some problems in viscous flow. Philos. Mag. (7) **35**, 250–262 (1944). [MF 10857]

For two dimensional slow steady movement of an incompressible viscous fluid (1) $\nabla^2\psi=-\zeta$, (2) $\nabla^2\zeta=0$, where ψ and ζ are the stream and circulation functions, respectively, subject to the condition (3) that the velocity vanishes on the boundaries. Under transformation to new coordinates $z=f(w)$, $z=x+iy$, $w=\xi+i\eta$, the solution of (1) and (2) can be made to depend on the determination of a certain analytic function $F(w)$. Specifically,

$$\zeta=4\left[\frac{F'(w)}{f'(w)}-\frac{F'(\bar{w})}{f'(\bar{w})}\right]. \tag{4}$$

Several interesting examples of flow in channels are obtained by taking various specializations of

$$z=B_0+B_1w+B_2e^{aw}+B_3e^{-aw}, \tag{5}$$

where B_3 is either 0 or B_2. In all cases treated the restriction is made that ψ depends on η alone and that the flow is symmetric with respect to $\eta=0$ so that ψ is odd in η. Then for each choice in (5) some consistent choices of F are obtained by inspection so that, in view of (4), (1) reduces to: (6) $d^2\psi/d\eta^2$ is a function of η alone. The complimentary solution of (6) is linear and involves only one essential constant since ψ and $\psi+C$ need not be distinguished. This constant and those in F are then to satisfy (3) for the choice of boundaries $\eta=\pm\eta_0$. *D. G. Bourgin* (Urbana, Ill.).

Aronovitch, G. V. Motion of a viscous incompressible fluid in a circular tube vibrating longitudinally. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 79–83 (1944). (Russian. English summary) [MF 11471]

Patry, J. Instabilité d'une rangée de tourbillons de long d'une paroi. Helvetica Phys. Acta **16**, 83–90 (1943).

The author considers the stability of a single row of vortices at a distance h from a solid wall in perfect fluid motion. The boundary condition is satisfied by means of the row of virtual vortices at a distance $-h$ from the wall. The stability of the so formed double row of vortices is investigated. This investigation differs from cases previously investigated because only symmetrical displacements of pairs of vortices are possible here. The author finds instability for all configurations and computes the path of a displaced vortex. A displacement normal to the wall is found to be the most "dangerous." In connection with the experimental work of L. Schiller on the origin of turbulence in channels, the following two effects are discussed. If a vortex is initially displaced away from the wall, it tends to combine with the vortices downstream and the row of vortices breaks off. If on the other hand the initial displacement is directed towards the wall this effect is not quite

reversed, due to the superimposed mean speed. This latter effect is taken as a qualitative reason for the occurrence of a comparatively small number of vortices in the entrance length of channels as observed by Schiller.

H. W. Liepmann (Pasadena, Calif.).

Basin, A. M. A new approximate method for the calculation of laminar boundary layer. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 14–17 (1943). [MF 11184]

The author proposes to use the family of velocity profiles

$$(1)\quad \frac{u(x,y)}{u(x,\delta)}=\sin\frac{\pi y}{2\delta}\left\{1+\frac{2\lambda}{\pi^2}\left(1-\sin\frac{\pi y}{2\delta}\right)\right\},\quad -\frac{\pi^2}{2}\leqq\lambda<\frac{\pi^2}{2},$$

(instead of Kármán-Pohlhausen's polynomial profiles) for the approximate calculation of steady boundary layer flows by using the momentum equation in the two-dimensional and axially-symmetrical cases. In (1) x and y are the distances along and perpendicular to the solid boundary, respectively, u is the velocity parallel to the x-direction, $\delta=\delta(x)$ is the thickness of the boundary layer, and λ is Pohlhausen's parameter $(\delta^2/\nu)(d/dx)u(x,\delta)$, where ν is the kinematic viscosity coefficient. This family of velocity profiles satisfies all the desirable boundary conditions of the von Kármán-Pohlhausen profiles, except the condition $\partial^2u/\partial y^2=0$ at $y=\delta$. [Reviewer's remark. Formula (1) reduces to Lamb's approximation for the case of zero pressure gradient and satisfies similar boundary conditions.] A simple approximate method of integrating the momentum equations is given.

It is stated that (1) is a good approximation not only in a region of accelerated flow near the point of minimum pressure, but also in a region of retarded flow up to the point of separation, $\lambda=\lambda_s=-\pi^2/2=-4.935$. This value of λ_s is compared with the experimental value $\lambda_s\cong-5.37$. The fact that the condition $(\partial^2u/\partial y^2)_{y=0}=0$ is satisfied for $\lambda=\lambda_s$ is given as the reason for the success of this family of profiles.

C. C. Lin (Pasadena, Calif.).

Littaye, Guy. Sur l'atomisation d'un jet liquide. C. R. Acad. Sci. Paris **217**, 340–342 (1943). [MF 11103]

The author continues his earlier experimental study of the breaking up of a liquid drop in a gas stream, based upon the relation [C. R. Acad. Sci. Paris **217**, 99–100 (1943)]

$$(1)\qquad (\rho(V-v)^2D)/T=\text{constant},$$

which gives the minimum relative velocity $V-v$ required to break up a drop of diameter D of a liquid of surface tension T, moving with velocity v in a stream of gas of density ρ flowing with velocity V. He considers a liquid drop accelerated from rest by a gas stream flowing with a velocity just sufficient to break it up. Assuming the duration of the splitting process to be equal to the period of vibration of the liquid drop, he finds that the drop splits after traveling a very small distance (about $0.6D$) and attaining only a very small fraction of the velocity of the gas stream (less than two per cent in the case of water drops in air). The effect of viscosity upon the period of oscillation has so far been neglected; the author proposes to return to this point later. From (1), it is concluded that there does not exist a minimum diameter to be attained by splitting up a liquid drop in a gas stream. Since the duration of splitting is very small, it is concluded that the mechanism is essentially associated with the wake and is not due to any centrifugal forces of the rotation of the liquid drop caused by turbulence in the gas stream.

C. C. Lin (Pasadena, Calif.).

Belenky, C. Z. Concerning the wave resistance of bodies in the super sound flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 84–87 (1944). (Russian. English summary) [MF 11472]

Lambin, N. V. Flow with separation past broken-line profiles of a grating. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 147–149 (1943). [MF 11060]

This is a rather concise outline of a calculation of the flow, with separation, through a two-dimensional infinite lattice of broken-line airfoils. A series of conformal transformations is used to transform the hodograph of the flow into a t-plane wherein the streamline bounding the airfoil-plus-wake contour appears as the coordinate axes of the first quadrant. The placing of certain singularities in the t-plane produces the desired flow pattern in the z-plane. No results are given in the present paper. It is stated that the problem has arisen in connection with the theory of supercavitating screw propellers. [Reviewer's note: it appears that the name "separation point" is erroneously applied to the stagnation point, and that flow is actually assumed to separate from the airfoil at its leading edge, as in certain analogous theories of single airfoils.]

W. R. Sears.

Gent, Betty L. Interference in a wind tunnel of regular octagonal section. Austral. Counc. Aeronaut. Rep. ACA—2, 8 pp. (1944). [MF 10900]

This paper determines mathematically the interference first on a small wing and then on one of finite span in a tunnel of regular octagonal section. For the tunnel shape considered it is not suitable to apply the method of images directly (as it is for the case of a rectangular or circular section). For the case of a small wing the octagon is first mapped into a half plane and the method of images is then applied. For the wing of finite span the factor of interference is deduced for both uniform and elliptic left distribution by mapping the section into a circle. The results obtained are very close to the corresponding results for a tunnel of circular section.

A. Gelbart (Syracuse, N. Y.).

Batchelor, G. K. Interference in a wind tunnel of octagonal section. Austral. Counc. Aeronaut. Rep. ACA—1, 11 pp. (1944). [MF 10901]

This paper gives a mathematical deduction of the interference on a model of small wing span suspended at the center of a closed tunnel of irregular octagonal section. The author follows essentially the procedure given earlier by Glouert [Aeronautical Research Committees, England, R. & M. no. 1566 (1933)] for a tunnel of rectangular section. The specific octagonal shape considered is that of the rectangular wind tunnels of the Council's Division of Aeronautics, Melbourne, of over-all dimensions 9 by 7 feet with right triangles of legs 7/4 feet cut away from each corner. It differs from the octagonal section-tunnel considered by Gent [cf. the preceding review] in that the corners cut away are essentially small. The induced velocity due to the walls is found to be a little greater than that in a rectangular section tunnel of the same over-all dimensions.

A. Gelbart (Syracuse, N. Y.).

Pérès, Joseph. Sur le calcul des effets d'ailerons pour l'aile de forme en plan elliptique. C. R. Acad. Sci. Paris **217**, 124–126 (1943). [MF 10637]

The author considers an elliptic wing twisted according to the formula (1) $\alpha=(\sin n\theta)/(\sin\theta)$, $0<\theta_1<\theta$; $\alpha=0$, $\theta_1<\theta<\pi$ (here α is the angle of twist, θ the usual angular

parameter for fixing the position of a wing-section, and n and θ_1 constants). He gives formulas for the spanwise lift distribution and discusses their usefulness for numerical computations. The twist formula (1) was introduced by H. Schmidt [Luftfahrtforschung **15**, 232–234, 246–255 (1938)] in order to investigate the effect of ailerons.

L. Bers (Providence, R. I.).

Laporte, Otto. Rigorous solutions for the spanwise lift distribution of a certain class of airfoils. Quart. Appl. Math. **2**, 232–250 (1944). [MF 11131]

The author treats the Prandtl theory of the wing of infinite span from the standpoint of Trefftz, that is, as a boundary-value problem of the third kind in a transverse u-plane ($u=y+iz$) far behind the wing. As in Trefftz' treatment [Z. Angew. Math. Mech. **1**, 206–218 (1921)] the trace of the wing in the u-plane is mapped on the unit circle in a t-plane. The author proceeds further to transform the t-plane into a Z-plane so chosen that the boundary condition, involving as it does the given planform function $c(y)$, becomes one with constant coefficients. The problem can then be treated by classical methods. This calculation is carried out for a special class of planforms given by

$$c(y)=c_0[(1-\eta^2)(1-\kappa^2\eta^2)]^{\frac{1}{2}},$$

where $c(y)$ is the chord, $\eta=2y/b$, b is the span, c_0 is a constant and κ is a parameter determining the taper as follows: (a) if $0\leqq\kappa\leqq 1$, wings of taper greater than the elliptic wing are obtained; (b) if $0\leqq\kappa\leqq i$, wings blunter than the elliptic. Case (a) is obtained by mapping the upper and lower halves of the unit circle on the top and bottom of a rectangle in the Z-plane ($Z=X+iY$), the vertical sides of the rectangle being mapped on slits extending into the circle. This transformation is $t=\sqrt{k}\,\mathrm{sn}\,(2K/\pi)Z$. Here k is the modulus of the elliptic function and K is the complete elliptic integral. Case (b) is obtained in an analogous manner by means of the transformation

$$t=i^{-1}\sqrt{k}\,\mathrm{sn}\,(2K/\pi)iZ.$$

In both cases the potential $\psi(X, Y)$ is finally obtained as an infinite series involving hyperbolic and trigonometric functions; in practice the convergence is found to be extremely rapid. Two limiting cases are considered separately: (i) the elliptic wing ($k=0$), (ii) the case of extreme taper ($k=1$). In case (i) the formulae are shown to agree with well-known results. Case (ii), in which the planform is formed by two parabolic arcs meeting at a sharp point, is treated independently, using the transformation $t=\tanh Z$.

Expressions are derived for the total lift and total induced drag. Numerical calculations have been carried out for the spanwise lift distribution for $k=1$ and for four other cases including the elliptic, each for three aspect ratios. The results are presented in graphical and tabular forms. In the case $k=1$ it was necessary to carry out some of the integrations graphically, and also to determine some roots of the transcendental equation

$$\sigma\cos\sigma+a\sin\sigma=0$$

for certain values of a.

W. R. Sears.

Rosenblatt, Alfred. Remarks on my paper "On the theorem of Kutta-Joukowsky" presented to the Academia Nacional de Ciencias de Lima on the 24th of April 1942. Actas Acad. Ci. Lima **6**, 130–131 (1943). (Spanish) [MF 9418]

This is a note concerning the present reviewer's review of the author's earlier paper [Revista Ci., Lima **44**, 259–292 (1942); these Rev. **4**, 177]. The author does not agree with the opinions expressed in the review.

W. R. Sears.

Jacob, Caïus. Sur le modérateur à ailettes. C. R. Acad. Sci. Paris **217**, 313–315 (1943). [MF 11100]

The author considers the flow in an infinite circular cylinder of radius R produced by a rotating system of n uniformly spaced blades. Each blade lies in a radial direction with edges at distances a and b from the axis of the cylinder ($0<a<b<R$), and extends to infinity in the axial direction. Assuming the motion to be a two-dimensional irrotational motion, with possible equal circulation around the blades, the author gives the complex potential of the flow relative to the rotating system of axes. The moment of force about the axis of symmetry and the kinetic energy of the system are considered. The limiting case $R\to\infty$ agrees with the results obtained previously by V. Valcovici [C. R. Acad. Sci. Paris **165**, 147–149 (1917); Ann. Sci. Univ. Jassy **10**, 5–24 (1920); Bull. Sci. École Polytech. Timişoara **2**, 199–237 (1929)] and W. Spannhake [for the case $a=0$, cf. Z. Angew. Math. Mech. **5**, 481–484 (1925)].

C. C. Lin.

Miles, John. The reflection of sound due to a change in cross section of a circular tube. J. Acoust. Soc. Amer. **16**, 14–19 (1944). [MF 10871]

Sound travels through a semi-infinite tube of radius $r=a_1$ for $z<\bar{z}$ and $r=a_2$ for $z>\bar{z}$. The sound pressure increments satisfy the wave equation. A plane harmonic wave is incident at one end. The method of treatment is standard. In cylindrical coordinates the reflected and transmitted waves permit of the usual orthogonal expansions involving Bessel functions. The boundary conditions are that the pressure is continuous at $\bar{z}$, $r<a_1$, and $\partial p/\partial z$ is 0 for the transmitted wave. This yields an infinite system of equations for the coefficients P_n of the reflected wave of the form

$$\sum_{n=0}^{\infty} A_{nj}P_n=A_{0j}D,$$

where D is a constant and A_{nj} is itself an involved infinite series of ratios of Bessel functions with constant argument determined by a_1 and a_2. The writer suggests an iteration method, essentially the method of finite segments. It seems to the reviewer that, quite apart from the desirability of showing convergence of the series for A_{nj} and the convergence of an iteration scheme, any "practical" solution of this sort would require the simplification of the expressions for A_{nj}; from the appearance of the series this seems a plausible undertaking.

D. G. Bourgin (Urbana, Ill.).

Theory of Elasticity

***Love, A. E. H. A Treatise on the Mathematical Theory of Elasticity.** Fourth Ed. Dover Publications, New York, 1944. xviii+643 pp. $3.95.

Reprint of the fourth (latest) English edition which appeared in 1927 at the Cambridge University Press.

***Leibenson, L. S. Variational Methods for the Solution of the Problems of the Theory of Elasticity.** Moscow-Leningrad, 1943. 288 pp. (Russian)

This book is based on the author's memoir under the same title published in 1940 [Trudi Centralnovo Aerohy-

drodynamicheskovo Instituta Imeni N. E. Joukowskovo, no. 495]. It is intended for a reader who is familiar both with the theory of elasticity and with the calculus of variations and who is primarily interested in applications. Accordingly the calculus of variations is treated as a course which yields effective methods for approximate solutions of boundary value problems arising in the theory of elasticity. Purely mathematical questions, like the convergence of the procedures used, are intentionally left aside. The content of the book is somewhat less general than the title might seem to indicate. For instance, comparatively little is said about dynamical problems. Finite difference methods are not mentioned. The most carefully written part of the book deals with the author's own work, which evidently has been stimulated by problems arising in aircraft engineering.

The first chapter contains a review of the fundamental equations of the theory of elasticity, a formal derivation of Lagrange's principle of virtual displacements and of Castigliano's principle, as well as a discussion of the general solutions of the elasticity equations in terms of arbitrary harmonic and biharmonic functions due to Galerkin, Neuber and Papkovitch. A survey of approximate variational methods for solving boundary value problems is given in chapters II and III, the latter dealing with the state of plane stress. Methods proposed by Ritz, Galerkin, Trefftz, Timoshenko, Kantorovitch, Papkovitch, and the author are described and in each case the conditions to be imposed on admissible functions are carefully noted. In the next two chapters these methods are applied to equilibrium and vibrations of elastic spheres, and to problems concerning strings, thin rods, membranes and thin plates.

Chapter VI deals with several approximate methods involving a relaxation of the rigorous boundary conditions. The author considers these methods as generalizations of the classical Saint-Venant principle. The examples given in this chapter are taken from the theory of membranes and thin plates. Here as well as in other sections the author draws general conclusions from the physical interpretation of the approximations used.

The remaining half of the book (chapters VII–XII) is devoted to torsion and bending of prismatic bodies. Numerous examples are given containing complete formal computations and numerical results. Often a problem is treated by several different methods. A comparison with rigorous solutions is given whenever possible. Aside from the rectangular and and triangular cross-sections the author considers mainly what he terms "airfoil profiles," that is, cross-sections bounded by one or two curves given by

$$y=a(x/b)^m[1-(x/b)^p]^q, \tag{1}$$

where a, b, m, p, q are constants. Anisotropic bodies are dealt with in the last two chapters.

No bibliography is included. *L. Bers.*

Pickett, Gerald. Application of the Fourier method to the solution of certain boundary problems in the theory of elasticity. J. Appl. Mech. **11**, A-176–A-182 (1944). [MF 11019]

The author shows how the method of orthogonal functions may be used to obtain solutions for stresses in rectangular prisms or circular cylinders for various boundary conditions. The calculations are based upon the linear equations of elasticity. The expansion coefficients in each problem are found from the solution of a system of infinitely many equations in infinitely many unknowns.

A. E. Heins (Cambridge, Mass.).

Süray, Saffet. Sur les lignes de tension principale dont une famille est constituée de lignes droites. Rev. Fac. Sci. Univ. Istanbul (A) **6**, 83–87 (1941). (French. Turkish summary) [MF 10816]

The following theorem is proved: a family of nonconcurrent, nonparallel straight lines cannot constitute one of the sets of principal stress trajectories of any homogeneous isotropic disc in two dimensional elastic equilibrium.

P. Nemenyi (Pullman, Wash.).

Charrueau, André. Sur les équilibres limites plans des milieux homogènes. C. R. Acad. Sci. Paris **217**, 311–313 (1943). [MF 11099]

The limiting state of equilibrium in a two dimensional elastic medium is governed by the equations

$$\frac{\partial N_x}{\partial x}+\frac{\partial T_{xy}}{\partial y}=X,\quad \frac{\partial T_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=Y, \tag{1}$$

$$\phi(N_x, N_y, T_{xy})=(N_x-N_y)^2+4T_{xy}^2-4a^2=0, \tag{2}$$

where (N, T) are the usual stress components, (X, Y) the body forces and a is the variable radius of Mohr's circle for a point (x, y) of the medium. In the case where

$$D(N_x, T_{xy})/D(x, y)\neq 0,$$

the author sets up a system of differential equations governing a correspondence from (x, y) space to (N_x, T_{xy}) space which is equivalent to (1) for a homogeneous medium.

D. L. Holl (Ames, Iowa).

Charrueau, André. Sur les équilibres limites plans des milieux hétérogènes. C. R. Acad. Sci. Paris **217**, 361–363 (1943). [MF 11104]

The author states that the analogous differential system constructed in the preceding review is adequate for a nonhomogeneous medium. An application is made to a case of variable body forces such as in gravitational attraction.

D. L. Holl (Ames, Iowa).

Ghosh, S. On the divergence of the solution of a problem of plane strain. Bull. Calcutta Math. Soc. **36**, 51–58 (1944). [MF 11350]

The paper is concerned with a well-explored problem of plane strain, namely, the determination of the stresses and displacements produced by a single force in an infinite elastic plane. The main result can be summed up by the following remark: the displacements which a single force produces at infinity do not vanish; if, however, a system of forces in equilibrium is considered instead of a single force, the displacements at infinity will vanish. *W. Prager.*

Bicadze, A. V. On local deformations in compressed elastic bodies. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 419–424 (1942). (Russian. Georgian summary) [MF 10327]

The author derives a formula for the pressure distribution along the line of contact between two plane elastic bodies pressed together. The boundaries L_1 and L_2 of the two bodies are given by sufficiently well-behaved even functions $y=f_i(x)$, $i=1, 2$, the x-axis being the common tangent to L_1 and L_2 at the original point of contact. The methods used are due to Muschelišvili. For the special case of two circular cylinders the author's result coincides with that previously obtained by L. Föppl [Z. Angew. Math. Mech. **16**, 165–170 (1936)]. *L. Bers* (Providence, R. I.).

Tolotti, Carlo. Über die Integration der Gleichungen des elastischen Gleichgewichts eines rechtwinkligen Parallelepipeds. Monatsh. Math. Phys. **50**, 176–204 (1941) = Ist. Naz. Appl. Calcolo (2) no. **110**. [MF 10481]

A rectangular parallelepiped of homogeneous isotropic material is considered. The surface forces are prescribed on the top and bottom faces, and either the surface force or the displacement is prescribed on the other four faces. Double Fourier transforms in the horizontal section of the parallelepiped are introduced, and the transformed equations of equilibrium are solved for the transforms of the displacements, the solutions being expressed in terms of definite integrals involving the prescribed boundary values mentioned above. *G. E. Hay* (Providence, R. I.).

Narodetzky, M. Z. A contact problem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 230–233 (1943). [MF 11067]

A force P is applied at the center of a circular disc exactly filling a circular hole in an infinite plate. It is assumed that there is no tangential stress on the arc of contact between the disc and plate. Using the method of functions of a complex variable as developed by Muschelišvili, the length of the arc of contact and the pressure along this arc are determined. *H. W. March* (Madison, Wis.).

Muschelišvili, N. I. On the problem of equilibrium of a rigid stamp on the boundary of an elastic half-plane with friction acting. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 413–418 (1942). (Russian. Georgian summary) [MF 10326]

The method described previously [same Bull. **2**, 873–880 (1941); these Rev. **6**, 28] is used to determine the stress distribution in an elastic half-plane ($y<0$) under the following boundary conditions. Along a segment L of the real axis the vertical displacement v satisfies the condition $v=f(x)+\text{const.}$, f being a given function. Along L the stress components satisfy the condition $X_y+kY_y=0$, k being a given positive constant. The remaining part of the real axis is stress free. Furthermore, the value of the integral $\int_L Y_y dx$ is given. This mathematical problem corresponds to the physical problem mentioned in the title. The function f is determined by the shape of the stamp; k is the coefficient of friction. *L. Bers* (Providence, R. I.).

Glagolev, N. I. Calculation of stresses due to the pressure of a system of rigid stamps. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 383–388 (1943). (Russian. English summary) [MF 11241]

Pucher, A. Rechteckplatten mit zwei eingespannten Rändern. Ing.-Arch. **14**, 246–266 (1943). [MF 11438]

This paper is concerned with the calculation of the stresses in transversely bent rectangular plates, two opposite edges of which are simply supported while the remaining two edges are clamped. The object of the paper is to make the known solution of the problem more readily applicable to special load conditions. For this purpose certain parts of the solution are tabulated once for all. *E. Reissner* (Cambridge, Mass.).

Sen, Bibhutibhusan. Note on the uniqueness of solution of problems of thin plates bent by normal pressures. Bull. Calcutta Math. Soc. **35**, 135–140 (1943). [MF 11011]

The author proposes to prove the uniqueness of the solution of the differential equation for the bending of a thin elastic plate with free or supported edges. The boundary conditions are, however, formulated in a way which is open to the following objection. The edge of the plate is taken as a coordinate curve of a system of orthogonal curvilinear coordinates. The boundary conditions used in the paper involve not only partial derivatives of the deflection along the tangent and normal of the edge, but also the curvature $(\partial\theta/\partial\nu)$ of the other coordinate curves at their points of intersection with the edge. This lets the boundary conditions depend on the choice of the family of curves in which the edge is embedded. For physical reasons this situation does not seem to be acceptable. *W. Prager.*

Bisshopp, K. E. Lateral bending of symmetrically loaded conical discs. Quart. Appl. Math. **2**, 205–217 (1944). [MF 11129]

The problem of the lateral bending of a conical disc considered as a circular plate of variable thickness leads to a special type of hypergeometric differential equation. By making certain transformations the solutions of this differential equation in the interval (0, 1) are expressed in terms of functions symmetrical with respect to the midpoint of this interval. To facilitate computation of these functions in the neighborhood of their singular points, each function is expressed as the sum of a properly chosen function having singularities of the same character at these points and a remainder which has uniformly small values throughout the interval. Tables are given for the coefficients needed in calculating the components of stress in the disc under prescribed loading symmetrical with respect to the center. The problem is closely related to that of the stress distribution in a rotating conical disc [K. E. Bisshopp, J. Appl. Mech. **11**, A-1–A-9 (1944)]. To illustrate an application of the theory the stress distribution in a large poppet valve is analyzed. *H. W. March* (Madison, Wis.).

Rosenblatt, Alfred. On the theory of elastic arches. I. Fundamental equations. Revista Ci., Lima **45**, 241–308 (1943). (Spanish) (1 plate) [MF 10079]

Ce mémoire développe le calcul des déformations et des tensions dans un arc circulaire. Dans ce but on calcule les premiers termes d'une fonction de Airy qui, dans l'hypothèse que les tensions soient indépendentes de la coordonnée angulaire, coincide avec un résultat déjà ancien de J. H. Mitchell. Dans un cas plus général on obtient des formules comparables avec certains résultats de Tiempe. On detaille finalement les calculs pour le cas d'un arc encastré aux extrémités et de petite épaisseur en comparant les resultats avec les formules adoptées par C. Guidi. *B. Levi.*

Chien, Wei-Zang. The intrinsic theory of thin shells and plates. III. Application to thin shells. Quart. Appl. Math. **2**, 120–135 (1944). [MF 10805]

In this paper the procedure used in part II [Quart. Appl. Math. **2**, 43–59 (1944); these Rev. **5**, 250] for thin plates is applied to thin shells. As before, a parameter ϵ is introduced; it is defined as the ratio of the average thickness of the shell to a selected lateral dimension. Since the shell is thin, $\epsilon \ll 1$. The quantities $b_{\alpha\beta}$ (α, $\beta=1$, 2), which are twice the coefficients in the second fundamental quadratic form of the unstrained middle surface of the shell, are assumed expressible as power series in ϵ, the exponent of the lowest power of ϵ being denoted by b. Thus, b is a measure of the curvature of the unstrained shell. When $b=0$, the curvature is comparable with the reciprocal of the selected lateral

dimension. When $b \geqq 1$, the former is much smaller than the latter.

As in part II, there are six fundamental equations for the six unknowns $p_{\alpha\beta}$, $q_{\alpha\beta}$; these unknowns represent, respectively, the extension and change of curvature in the middle surface, and, once they are known, the stress and strain in the shell follow readily. Solutions are sought in the form of power series in ϵ, the exponents of the lowest powers of ϵ being denoted by p and q, respectively. It is found that the fundamental equations take on 35 distinct forms, depending on the values of b, p, q. These forms are classified. Of them, eight correspond to the case $b=0$ mentioned above, and eleven are equivalent to problems in thin plates discussed in part II. *G. E. Hay* (Providence, R. I.).

Morkovin, Vladimir. Membrane stresses in shells of constant slope. Quart. Appl. Math. **2**, 102–112 (1944). [MF 10803]

A surface of constant slope is generated by a straight line sliding along a plane curve, at right angles with the tangent to the curve and enclosing a given angle with its binormal. For thin elastic shells of this type the author obtains explicitly the stress resultants and displacements of "membrane" theory for the case that the external load does not vary in the direction of the generators of the shell.

E. Reissner (Cambridge, Mass.).

Sokolovsky, W. W. Author's remarks on the article "Equations of Momentless Shells." Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 88 (1944). (Russian. English summary) [MF 11473]

Correction to the paper in the same journal **7**, 57–64 (1943); these Rev. **5**, 139.

Novojilov, V. General theory of stability of thin shells. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 316–319 (1941). [MF 10989]

The differential equations of this theory are given and their derivation is sketched. References to earlier work are not included. *E. Reissner* (Cambridge, Mass.).

Novojilov, V. and Finkelstein, R. Concerning inaccuracies in the theory of shells due to the Kirchoff hypothesis. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 331–340 (1943). (Russian. English summary) [MF 11236]

Deuker, Ernst-August. Zur Stabilität der elastischen Schalen. II. Z. Angew. Math. Mech. **23**, 169–179 (1943). [MF 10180]

[Part I appeared in the same Z. **23**, 81–100 (1943); these Rev. **5**, 195.] The author continues his investigations in the theory of thin elastic shells [cf. also Deutsche Math. **5**, 546–562 (1941); these Rev. **3**, 95]. In the present paper the problem of the stability of the closed spherical shell under uniform external pressure is treated in a very detailed way. The results are essentially the same as those of the well-known linear theory. *J. J. Stoker* (New York, N. Y.).

Lourie, A. I. Equilibrium of an elastic symmetrically loaded spherical shell. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 393–404 (1943). (Russian. English summary) [MF 11259]

Djanelidze, G. J. Application of the variation method to the theory of thin-walled elastic bars developed by V. Z. Vlasov. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 455–462 (1943). (Russian. English summary) [MF 11264]

1. Expressions for the deformation of the central surface of a cylindrical shell are deduced, assuming its cross-section to be undeformable. 2. The expression for the potential energy of a curved and contorted cylindrical shell with an undeformable cross-section is constructed. 3. Differential equations are set up for equilibrium in displacements and for conditions at the ends of the bar similar to the equations of Euler-Lagrange and for so-called natural boundary conditions of the variation problem concerning the potential energy of the system. *Author's summary.*

Shapiro, G. S. An infinite hollow circular cylinder loaded by an external pressure on a finite range of its length. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 379–382 (1943). (Russian. English summary) [MF 11240]

Gorgidze, A. Ja. and Ruchadze, A. K. On second-order effects in the bending of a circular cylinder. I, II. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 397–404, 491–498 (1941). (Russian. Georgian summary) [MF 10305]

Part I. An isotropic circular cylinder with fixed base and force-free sides is subjected to transversal forces acting upon the free end face; they are statically equivalent to a bending and a torsion couple. The stress $(X_z, \cdots, Y_z)$ and displacement (u, v, w) distributions of the linear theory are well known; in this paper the terms which are quadratic in the torsion and bending strain constants τ and ν are determined. The treatment is based on the general theory of finite deformations by F. D. Murnaghan [Amer. J. Math. **59**, 235–260 (1937)], carried to the order of approximation which involves the second and third powers of the strain invariants in the strain energy function. This approximation involves three elastic constants and the two isotropic constants λ, μ of the linear theory. Inasmuch as numerical values of the former are not known for any material, a "partial" nonlinear treatment involving λ and μ only is called for; the third-order terms in the strain energy are dropped, but the quadratic strain terms arising from the noninfinitesimal strain variations are retained. The corresponding expressions on which this paper is based are, however, at variance with Murnaghan's results; no derivation is given, reference being made to a Russian paper by N. V. Zvolinski and P. M. Risz [Trans. Acad. Sci. **22**, no. 2–3, 101 (1938)] which was not accessible to the reviewer. The procedure is straightforward; the determination of the stress distributions X_z', $\cdots$, Y_z'; $\cdots$; X_z''', $\cdots$, Y_z''' which correspond to u_1, v_1, w_1; $\cdots$; u_3, v_3, w_3, respectively, involves the solution of three sets of linear problems with known solutions; each set comprises a triplet of first-order partial differential equations, one of ordinary, and one of second-order equations, representing the equilibrium, surface and compatibility conditions, respectively; in keeping with the order of approximation, the transformation of the surface normals in the second triplets involves linear theory only. The resulting 3×9 equations are written out. The equations for the stresses x_z''', $\cdots$ are exceedingly complicated and cover two pages.

Part II. The stress distributions associated with τ^2 and $\tau\nu$ are determined. The procedure is straightforward, the

functions involved being split up into rational components and general harmonic and biharmonic functions, which in turn come out rational (of second and third order, respectively, for the stress components). The final expressions in terms of the original constants are extremely lengthy. The corresponding displacement distributions which follow directly are not written out. The second-order solution obtained thus far is complete for the case of large torsion in combination with moderate bending and gives, in particular, the first-order torsion-bending interaction term. It is indicated that the method can be modified for the treatment of general cross-sections. *H. G. Baerwald.*

Gorgidze, A. Ja. and Ruchadze, A. K. On second-order effects in the bending of a circular cylinder. III. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 221–228 (1942). (Russian. Georgian summary) [MF 10325]

[Cf. the preceding review.] This part is concerned with the stress distribution associated with the term ν^2. The results are again rational in the coordinates (of fourth order) and extremely lengthy due to elementary but involved dependence on the constants. The solution of the second-order nonlinear torsion-bending problem is thus considered complete inasmuch as the addition of the solution of a linear Saint-Venant problem would satisfy boundary conditions at the end face corresponding to a given system of forces. *H. G. Baerwald* (Cleveland, Ohio).

Ruchadze, A. K. The bending of a stretched prismatic rod by a transversal force. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 609–616 (1941). (Russian. Georgian summary) [MF 10306]

A system of forces statically equivalent to a longitudinal and a bending force act upon the free end face of an isotropic prismatic rod clamped at the base and free at the sides. The couple is assumed to be of moderate magnitude, that is, within the limits of the linear theory, while the longitudinal force is large. Denote by α, τ, ν the strain constants corresponding to normal, torsion and bending strain, respectively. The solution of the linear theory contains only terms which are linear in these constants. The author studies the terms containing α^2, $\alpha\tau$ and $\alpha\nu$, which give the next approximation for large α; the two interaction coefficients represent the additional longitudinal displacements resulting from superposition of (small) transversal forces upon a large longitudinal strain. The basis and method of attack are those of the papers reviewed above [there is the same moot point regarding the basic nonlinear stress-strain relations]. *H. G. Baerwald* (Cleveland, Ohio).

Barten, H. J. On the deflection of a cantilever beam. Quart. Appl. Math. **2**, 168–171 (1944). [MF 10810]

Formulas are given for calculating the deflection of a cantilever beam of such low stiffness that large deflections are possible in the elastic range. For such deflections the square of the first derivative cannot be neglected in the expression for the curvature. The analysis leads to elliptic integrals as in the case of the elastica. *H. W. March.*

Schevchenko, K. N. External pulling stress of a bar during rolling. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 389–392 (1943). (Russian. English summary) [MF 11242]

***Brodeau, André. Poutres minces. Flexion, torsion, aplatissement.** Actual. Sci. Ind., no. 920. Hermann et Cie., Paris, 1942. 70 pp.

The author proposes to study the bending deformations of beams of thin-walled sections using the concept of flattening forces. These forces are defined in the following manner. The usual flexural stresses, when considered as acting on the deformed beam, do not constitute an equilibrium system but furnish resultant forces which act along the principal normals of the bent fiber and are directed towards the neutral layer of the beam. For solid sections the secondary stresses corresponding to these forces can be neglected. In the case of thin-walled sections, however, these forces may produce a considerable flattening of the cross-section and, eventually, cause the beam to fail. While the use of this concept of flattening forces may be justified as one step in a series of successive approximations, the author takes it for granted that no further steps are required in order to obtain a correct picture of the bending deformations and makes no attempt at investigating the convergence problem associated with his procedure. *W. Prager.*

Mandel, Jean. Sur l'instabilité d'une tige par torsion avec ou sans effort longitudinal. C. R. Acad. Sci. Paris **217**, 363–364 (1943). [MF 11105]

Fouché, Marcel. Verges courbes, verges coudées vibrant dans leur plan. Rev. Fac. Sci. Univ. Istanbul (A) **8**, 160–192 (1943). (French. Turkish summary) [MF 10933]

The author records the results of determining experimentally the frequencies of vibration of various forms of rods. These include circular arcs of constant circular section with clamped ends for variable radius and constant length and also for constant radius of arc but with variable arc length. Other forms considered include the tuning fork type consisting of a semicircle with rectilinear tines, or two rectilinear tines joined at an angle, or a single rectilinear segment and circular arc. *D. L. Holl* (Ames, Iowa).

Platrier, Charles. Détermination de la fibre moyenne déformée d'une poutre. C. R. Acad. Sci. Paris **217**, 283–285 (1943). [MF 11114]

A rod of cross section S (not necessarily uniform, but doubly symmetrical with respect to orthogonal axes through the centroid) is acted upon by a system of forces which are equivalent to components X, Y, Z at the centroid and a couple with components L, M, N. The author finds expressions for the changes in curvature and in twist of a central fiber in terms of (X, Y, Z), of (L, M, N), of moduli (E, G, σ), and of a harmonic function ψ whose normal derivative is $\partial\psi/\partial n = ly - mx$ on the boundary of S with (l, m) direction cosines. *D. L. Holl* (Ames, Iowa).

Goodier, J. N. A theorem on the shearing stress in beams with applications to multicellular sections. J. Aeronaut. Sci. **11**, 272–280 (1944). [MF 10771]

The author defines pure bending of a cantilever, that is, bending free from torsion, by the condition that the average value of the rate of axial rotation $\partial w_z/\partial z$ over the beam section shall vanish. This shows that the shear center depends upon Poisson's ratio σ, and as a result it is shown that the line integral $\int \tau ds$ around a closed circuit does not in general vanish. Here τ is the resultant shear and the integral is taken not necessarily around the boundary but around any closed contour. Application is made to the calculation of flexural shearing stress in wing beams of multicellular

section and the results for certain rectangular sections are compared with results obtained by employing other defining criteria for the shear center [see, e.g., W. R. Osgood, J. Appl. Mech. **10**, A-62–A-64 (1943); these Rev. **5**, 84].

D. L. Holl (Ames, Iowa).

Goodier, J. N. On combined flexure and torsion, and the flexural buckling of a twisted bar. Quart. Appl. Math. **2**, 93–101 (1944). [MF 10802]

When a straight slender bar is twisted, the straight form becomes unstable at a certain value of the twisting couple, and the centerline of the bar becomes a space curve. Let EI_1, EI_2, GC be flexural and torsional rigidities, u, v components of deflection parallel to the principal axes of the section, and β the torsional rotation. The author observes that, if the strain energy is represented in the accepted form of small bending and torsion of slender bars:

$$\tfrac{1}{2}\int[EI_1(u'')^2+EI_2(v'')^2+GC(\beta')^2]dz,$$

then the energy method fails to give the correct known differential equations of the theory. This situation is remedied by the author by taking the strain energy in the form

$$\tfrac{1}{2}\int[EI_1k_1^2+EI_2k_2^2+GC\tau^2]dz$$

and by developing the curvature and torsion terms up to quantities of the next higher order. They are,

$$k_1=-u''+\beta v'', \quad k_2=v''+\beta v', \quad \tau=\beta'+u''v'.$$

E. Reissner (Cambridge, Mass.).

Van Wyk, Juan C. Simplified technical solution of the problem of torsion. Bol. Fac. Ingen. Montevideo **2**, (Año 6), 1–22 (1941). (Spanish) [MF 10836]

The author, apparently guided by the elementary treatment of torsion of an elliptic cylinder [see, for instance, C. Bach, Elasticität und Festigkeit, Berlin, 1894, 2nd ed., pp. 161–163], suggests a similar method for other cross sections which occur in technical applications. It is based on an empirical subdivision of the cross section into ring-shaped areas, the shearing stress in each of such areas being assumed constant. The law of variation of the shearing stress from area to area is also assumed in an intuitive manner. The maximum shearing stress follows then by an elementary analysis of equilibrium of forces.

I. Opatowski (Chicago, Ill.).

Reissner, Eric and Sagoci, H. F. Forced torsional oscillations of an elastic half-space. I. J. Appl. Phys. **15**, 652–654 (1944).

Torsional displacements, such as those that would result from the application of the load by means of a rigid disk, are prescribed over a circular area of an elastic half-space. E. Reissner [Ing.-Arch. **8**, 229–245 (1937)] solved a related problem in which the shear stress is prescribed over a circular area, the shear stress increasing linearly from the center to the circumference. In the present paper it is shown that an explicit solution of the appropriate boundary value problem can be obtained by introducing a system of oblate spheroidal coordinates. For the static case an expression is obtained for the angle of rotation in terms of the applied torque and for the distribution of shear under the rigid disk. [See the following review.]

H. W. March (Madison, Wis.).

Sagoci, H. F. Forced torsional oscillations of an elastic half-space. II. J. Appl. Phys. **15**, 655–662 (1944).

In this paper, a continuation of the paper reviewed above, the prescribed torsional displacements over a circular area on the surface of an elastic half-space vary periodically with the time. The solution of the appropriate wave equation is found in terms of functions of spheroidal coordinates. Expressions are obtained for the displacement and the shear stress at any point of the surface and for the applied torque in terms of the angle of rotation of the circular area over which the displacement is prescribed. *H. W. March.*

Lehr, Georges. Sur la forme de l'équation générale aux fréquences propres des arbres vibrant en torsion. C. R. Acad. Sci. Paris **217**, 285–287 (1943). [MF 11115]

This note gives some results concerning the natural frequencies of torsional vibration of a system consisting of N masses, having moments of inertia $m_1, m_2, \cdots, m_N$, connected by shafts having stiffnesses $c_1, c_2, \cdots, c_{N-1}$. It is shown that the equation determining the frequencies can be obtained by a process other than the usual one of expanding a determinant. It is also shown that when the m's are equal and the c's are equal the roots of the equation can be obtained by a simple graphical process. The latter result is not new. *L. A. MacColl* (New York, N. Y.).

Tumarkin, S. Methods of stress calculation in rotating disks. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1064**, 58 pp. (4 plates) (1944). [MF 10932]

Translation of the report No. 262 of the Central Aero-Hydrodynamical Institute, Moscow, 1936.

Gurin, A. I. Dynamical stability of a flexible shaft fitted with a disc. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 310–315 (1943). (Russian. English summary) [MF 11232]

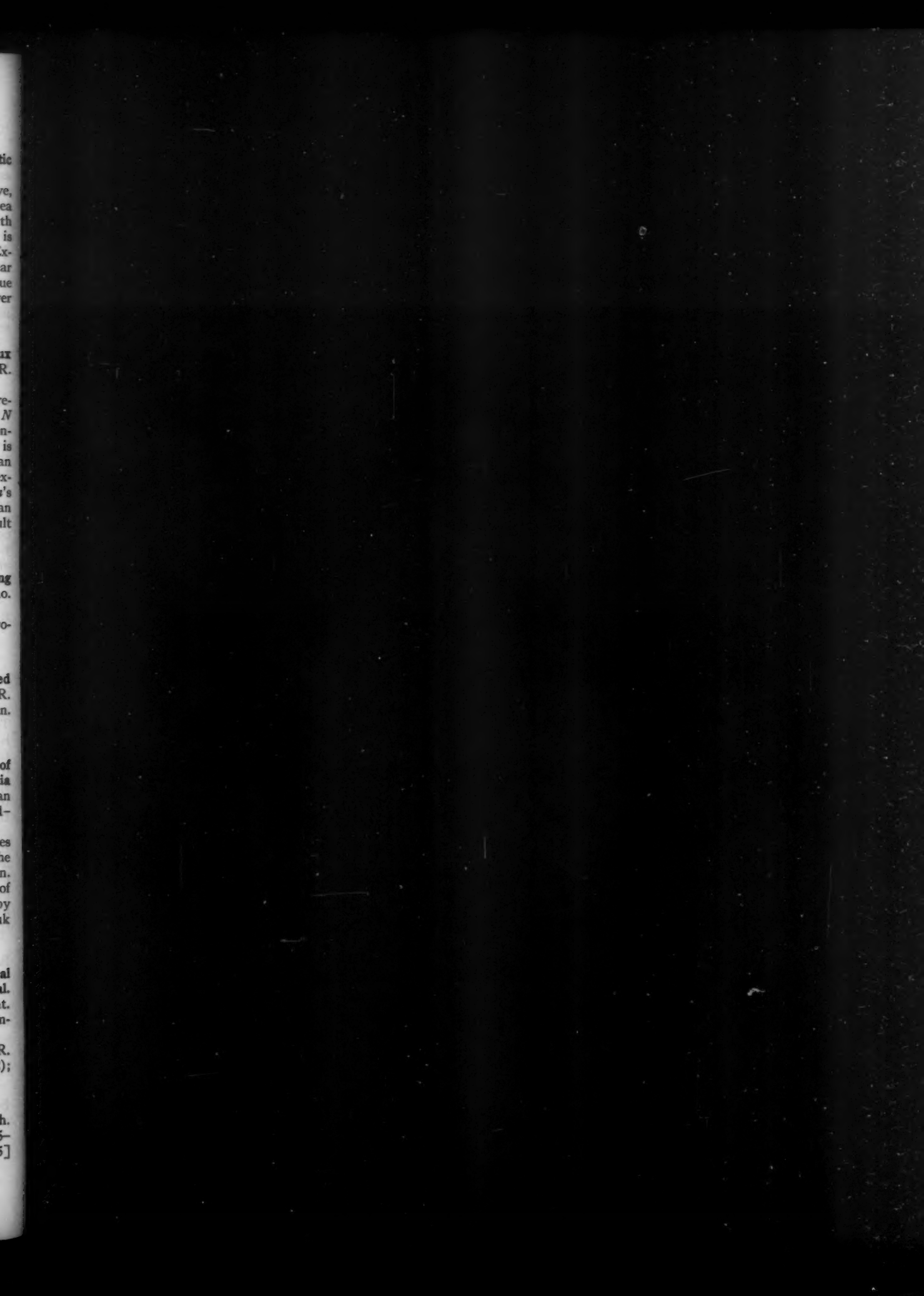
Berberjan, L. G. On the problem of the propagation of surface waves on the boundary of two distinct media sliding one over the other. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 321–326 (1941). (Russian) [MF 10304]

This note deals with the propagation of disturbances along the common boundary of two elastic half-spaces. The author discusses the roots of a certain algebraic equation. Upon the location of these roots depends the possibility of occurrence of surface waves of a type discovered by Kupradze and Sobolev [Trudi Seismol. Inst. Akad. Nauk SSSR, no. 10 (1932)]. *L. Bers* (Providence, R. I.).

Sokolovsky, W. W. Elastico-plastic state of a cylindrical tube yielding with a strain-hardening of material. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 273–292 (1943). (Russian. English summary) [MF 11229]

An extensive discussion of the results announced in C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 160–165 (1942); these Rev. **5**, 84.

Galerkin, B. G. The stability of a waterpipe. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 325–330 (1943). (Russian. English summary) [MF 11235]



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